Mathematical problems of General Relativity
Lectures 4-5

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Outline

1. Energy and momentum in General Relativity
   - Basic definitions
   - Positivity of energy
2. Symmetries and the initial value problem
3. Epilogue: formulation of the Cauchy problem for the Einstein field equations

Outline

Energy and momentum in theories of gravitation: basic issues

The equivalence principle:

- It is a well know problem in General Relativity that energy and momentum of the gravitational field cannot be localised.
- This is a direct consequence of the **equivalence principle**.
- As a consequence one cannot define, for example, a density of energy for the gravitational field.
- However, it is still possible to define some **global conserved quantities** which, in turn, can be interpreted as the total energy of a gravitating system.
- These quantities behave in a similar way to electromagnetic charges —that is, they take the form of volume integrals which are transformed into surface integrals.

The ADM energy and momentum (I)

The definition:

- In what follows let $(\mathcal{S}, h_{ij}, K_{ij})$ denote an initial data set for the vacuum Einstein field equations —i.e. they satisfy the constraints.
- Let x^α denote **asymptotically Cartesian coordinates** —i.e. a system of coordinate for which $h_{\alpha\beta}$ is $\delta_{\alpha\beta}$ to first order.
- One defines the **ADM energy** as the surface integral

$$E = \frac{1}{16\pi}\int_{\mathcal{S}_\infty} (\partial^\beta h_{\alpha\beta} - \partial_\alpha) n^\alpha \mathrm{d}S, \qquad h \equiv h_{\alpha\beta}\delta^{\alpha\beta}.$$

where S_∞ denotes the **sphere at infinity**, and n^α is the outward pointing normal to the sphere. Similarly, the **ADM momentum** is given by

$$p^\alpha = \frac{1}{8\pi}\int_{\mathcal{S}_\infty} (K^\alpha{}_\beta - K\delta^\alpha{}_\beta) n^\alpha \mathrm{d}S.$$

The ADM energy and momentum (II)

Coordinate independence:

- The expressions can be shown to be coordinate independent.
- In particular, a change to another asymptotically Cartesian system gives the same ADM mass and momentum.
- The energy E and the momentum p^α are the components of a 4-dimensional vector (4-vector) —the **ADM 4-momentum vector**:

$$p^\mu = (E, p^\alpha).$$

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Finiteness:

- If one has an initial data set $(\mathcal{S}, h_{ij}, K_{ij})$ satisfying

$$h_{\alpha\beta} - \delta_{\alpha\beta} = O(1/r), \qquad K_{ij} = O(1/r^2),$$

then one can readily verify that

$$E < \infty, \qquad p^\alpha < \infty.$$

- The verification of the above statement for p^α makes use of the constraint equations.

The ADM energy and momentum (III)

Some intuition: the Schwarzschild spacetime

- In order to obtain intution into the content of the ADM energy and momentum, it is convenient to evaluate them on the Schwarzschild spacetime.
- Make use of the time symmetric hypersurface given in standard coordinates by constant t.
- As already seen, for this hypersurface it has been seen that $K_{ij} = 0$. Moreover, one has that

$$h_{\alpha\beta} = \left(1 + \frac{m}{2r}\right)^4 \delta_{\alpha\beta}.$$

- A calculation then shows that

$$E = m, \qquad p^\alpha = 0.$$

- The ADM energy of the time symmetric slice of the Schwarzschild spacetime coincides with its mass parameter.

Conservation of the ADM energy and momentum (I)

Intuition:

- As p^α provides a measure of the total energy of a gravitating system, it is natural to expect that its components satisfy some sort of conservation behaviour.

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Showing the conservation:

- Consider an evolution off the hypersurface $\mathcal{S}$ such that

$$\alpha = 1 + O(1/r), \qquad \beta^\alpha = O(1/r).$$

  The latter corrresponds to an evolution into nearby hypersurfaces $\mathcal{S}$ which are essentially a time translation at infinity. From the above assumptions it follows that $\mathcal{L}_\beta g_{\mu\nu} = O(1/r^4)$.

- One then computes $\partial_t E$ to obtain

$$\partial_t E = \int_{\mathcal{S}_\infty} (\partial_t \partial^\beta h_{\alpha\beta} - \partial_t \partial_\alpha h) n^\alpha \mathrm{d}S.$$

- Using the ADM evolution equations one can readily verify by inspection that

$$\partial_t \partial^\beta h_{\alpha\beta} - \partial_t \partial_\alpha h = O(1/r^3) \Longrightarrow \partial_t E = 0.$$

Conservation of the ADM energy and momentum (II)

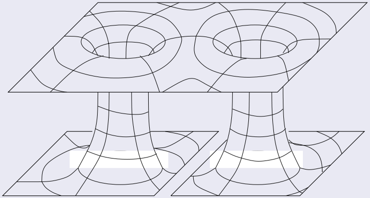
Showing the conservation:

- A similar argument shows that $\partial_t p^\alpha = 0$. Thus, indeed, the components of p^μ are conserved, at least for evolutions which behave as a time translation at infinity.

Some further remarks (I)

$$E = \frac{1}{16\pi}\int_{\mathcal{S}_\infty}(\partial^\beta h_{\alpha\beta} - \partial_\alpha)n^\alpha \mathrm{d}S, \qquad h \equiv h_{\alpha\beta}\delta^{\alpha\beta}$$

- Observe that although E measures thetotal energy contained in $\mathcal{S}$, it is expressed as a surface integral on an asymptotically end.
- An **asymptotic end** is a subset $\mathcal{U} \subset \mathcal{S}$ which is diffemorphic (i.e. it can be identified) with the complement of a ball in $\mathbb{R}^3$. That is, $\mathcal{U} \approx \mathbb{R}^3 \setminus B_R$.
- One can have several asymptotic ends in $\mathcal{S}$ as in the case of Brill-Lindquist data. **Each asymptotic end has its own wn ADM mass!**



Some further remarks (II)

Coordinate invariance:

- On each asymptotic end one requires for suitably large R in $\mathbb{R}^3 \setminus B_R$ that the metric approaches the flat metric δ_{ij} —**asymptotic Euclidean data**. For example
$$h_{\alpha\beta} = \delta_{\alpha\beta} + O(r^{-1}).$$
- The coordinates rendering the above expression are called **asymptotically Euclidean coordinates**.
- The formula of the ADM mass can be shown to be independent of the particular choice of asymptotically Euclidean coordinates **[Bartnik, 1982]**.

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Positivity of energy in (pseudo) Newtonian theories (I)

Basic intuition:

- On intutitive grounds one would expect the ADM 4-momentum to satsify some **positivity properties**.
- This is not at all obvious from the definitions in terms of surface integrals of the ADM energy and momentum.

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A toy model:

- Let ϕ denote the **gravitational potential** and let ρ denote the **density of matter**.
- In physically reallistic situations one expects ρ to be a function of **compact support** —that is, it vanishes outside a compact set. This requirement fits naturally with the notion of isolated system.
- The gravitational potential is related to the density via the Poisson equation

$$\Delta\phi = 4\pi G\rho.$$

- The total mass of the system is just the integral of the density over the whole space:

$$m = \int_{\mathbb{R}^3} \rho \mathrm{d}^3 x < \infty.$$

Positivity of energy in (pseudo) Newtonian theories (II)

Computation of the total energy:

- The total energy of the system is then given (using special relativistic arguments) by

$$\begin{aligned} E_{total} &= mc^2 + E_{grav} \\ &= c^2 \int_{\mathbb{R}^3} \rho \mathrm{d}^3 x + \frac{1}{2}\int_{\mathbb{R}^3} \rho\phi \mathrm{d}^3 x \\ &= c^2 \int_{\mathbb{R}^3} \rho \mathrm{d}^3 x + \frac{1}{8\pi G}\int_{\mathbb{R}^3} \phi\Delta\phi \mathrm{d}^3 x \\ &= c^2 \int_{\mathbb{R}^3} \rho \mathrm{d}^3 x - \frac{1}{8\pi G}\int_{\mathbb{R}^3} |\nabla\phi|^2 \mathrm{d}^3 x. \end{aligned}$$

- In the last equation the second term is negative so that the energy is, in principle, **not bounded from below**.
- This is a problem, as it could mean one could extract and infinite amount of energy out a gravitating system.
- General Relativity deals with this problem by **postulating the Universality of Gravity** —that is, the fact that gravity can act as source of itself.

Energy positivity in General Relativity (I)

Observation:

- The universality of Gravity in General Relativity ensures the positivity of the energy —the so called **mass positivity theorem**, Schoen & Yau 1979-1981.

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Theorem

Consider a time symmetric initial data set for the vacuum Einstein field equations —i.e. $K_{ij} = 0$. *Assume that* $\mathcal{S} \approx \mathbb{R}^3$ *with*

$$h_{\alpha\beta} - \delta_{\alpha\beta} = O(1/r),$$

and that $r_{ij}\lambda^i\lambda^j \geq 0$ *for* $\lambda^i \neq 0$. *Then*

$$E > 0.$$

Rigidity part of the posivity of energy theorem

Remark:

- The theorem has also a **rigidity part**:
  - If the energy vanishes and the hypersurface is regular, then the hypersurface is flat.
- This implies that vacuum cannot gravitate —notice that this result depends strongly on the boundary conditions being used.

Conformally flat data (I)

Observation:

- A simple proof of the positivity of the mass can be given in the case of time symmetric ($K_{ij} = 0$), conformally flat data:

$$h_{ij} = \vartheta^4 \delta_{ij}.$$

The Hamiltonian constraint

In this case the Hamiltonian constrain takes the form

$$r = \rho$$

with ρ the energy density. It follows then that

$$\Delta\vartheta + \frac{1}{8}\vartheta^5 \rho = 0.$$

Asymptotic flatness requires

$$\vartheta = 1 + u, \qquad u = O(r^{-1}), \qquad \partial_\alpha u = O(r^{-2}).$$

Conformally flat data (II)

Observing that

$$\partial_r = n^\alpha \partial_\alpha = \frac{x^\alpha}{r}\partial_\alpha,$$

and recalling that

$$E = \frac{1}{16\pi}\int_{\mathcal{S}_\infty} (\partial^\beta h_{\alpha\beta} - \partial_\alpha) n^\alpha \mathrm{d}S,$$

one finds that

$$E = -\frac{1}{2\pi}\int_{\mathcal{S}_\infty} \partial_r \vartheta \mathrm{d}S = -\frac{1}{2\pi}\int_{\mathcal{S}_\infty} \partial_r u \mathrm{d}S.$$

A calculation then shows that

$$\Delta\vartheta + \frac{1}{8}\vartheta^5\rho = 0 \Longleftrightarrow \frac{1}{8}\rho = -\partial^\alpha\left(\frac{\partial_\alpha\vartheta}{\vartheta^5}\right) - 5\frac{\partial_\alpha\vartheta\partial^\alpha\vartheta}{\vartheta^6}.$$

Conformally flat data (III)

Integrating over $\mathbb{R}^3$, using the Gauss theroem, one finds

$$\begin{aligned} 0 \leq \frac{1}{2\pi}\int_{\mathbb{R}^3}\left(\frac{1}{8}\rho + \frac{5|\partial\vartheta|^2}{\vartheta^6}\right)d^3x &= \frac{1}{2\pi}\int_{\mathbb{R}^3}\partial^\alpha\left(\frac{\partial_\alpha\vartheta}{\vartheta^5}\right)d^3x \\ &= -\frac{1}{2\pi}\int_{\mathcal{S}_\infty}\frac{\partial_\alpha\vartheta}{\vartheta^5}n^\alpha dS \\ &= -\frac{1}{2\pi}\int_{\mathcal{S}_\infty}\partial_\alpha\vartheta n^\alpha dS = E \end{aligned}$$

as $\vartheta \to 1$ as $r \to \infty$. Hence

$$m \geq 0.$$

Brill-Lindquist data

For Brill-Lindquist data one has

$$\mathcal{S} = \mathbb{R}^3 \setminus \{i_1, i_2\},$$

and that

$$\vartheta = 1 + \frac{m_1}{2r_1} + \frac{m_2}{2r_2}, \qquad m_1, m_2 \geq 0.$$

In this case one has 3 masses:

$$E_0 = m_1 + m_2,$$
$$E_1 = m_1 + \frac{m_1 m_2}{L},$$
$$E_2 = m_2 + \frac{m_1 m_2}{L},$$

with L the Euclidean distance between i_1 and i_2. The terms $m_1 m_2/L$ are interaction terms.

Outline

A simple example: slices of Minkowski spacetime

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Local solution to the problem of characterisation of initial data:

- The pair (h_{ij}, K_{ij}) of symmetric tensors corresponds (locally) to the first and second fundamental form of a slice $\mathcal{S}$ in Minkowski spacetime if and only if

$$D_{[i}K_{j]l} = 0,$$
$$r_{ijkl} = -2K_{k[i}K_{j]l}.$$

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A global characterisation (Schoen & Yau):

- The pair, (h_{ij}, K_{ij}), of smooth asymptotically Euclidean symmetric tensors corresponds (locally) to the first and second fundamental form of a slice $\mathcal{S}$ in Minkowski spacetime if and only if its ADM mass is zero.

Encoding symmetries in the initial data

The problem:

An issue which often ariseses in the analysis of the Cauchy problem for the Einstein field equations is that of encoding in the initial data the fact that the resulting spacetime will have a certain symmetry —i.e. a Killing vector. This naturally leads to the notion of **Killing initial data (KID)**.

Some consequences of the Killing equation (I)

Idea:

In order to analyse the question raised in the previous paragraph, it is necessary to first consider some consequences of the Killing equation

$$\nabla_a \xi_b + \nabla_b \xi_a = 0.$$

An integrability condition:

Applying ∇^a to the above equation and commuting covariant derivatives one finds that

$$\begin{aligned} 0 &= \nabla^a \nabla_a \xi_b + \nabla^a \nabla_b \xi_a \\ &= \Box \xi_b - R^c{}_b \xi_c, \end{aligned}$$

where it has been used that $\nabla^a \xi_a = 0$. Accordingly, in vacuum one has that a Killing vector satisfies the Killing vector equation

$$\Box \xi_a = 0.$$

In what follows a ξ_a satisfying the wave equation (24) will be called a **Killing vector candidate**.

Some consequences of the Killing equation (II)

A wave equation for the Killing equation

Now, in what follows let

$$S_{ab} \equiv \nabla_a \xi_b + \nabla_b \xi_a,$$

and compute $\square S_{ab}$. To this end notice that commuting covariant derivatives and using that by assumption $R_{ab} = 0$ and $\nabla^e R^f{}_{bea} = 0$ one has

$$\begin{aligned} \square S_{ab} &= R^e{}_a{}^f{}_b \nabla_f \xi_e + R^e{}_a{}^f{}_b \nabla_e \xi_f + \nabla_a \square \xi_b + \nabla_b \square \xi_a \\ &= R^e{}_a{}^f{}_b S_{ef} + \nabla_a \square \xi_b + \nabla_b \square \xi_a. \end{aligned}$$

The KID conditions

Obtaining conditions on the initial data:

Assume that one has a vector ξ^a satisfying $\Box\xi_a = 0$. One has then that

$$\Box S_{ab} - R^e{}_a{}^f{}_b S_{ef} = 0.$$

If initial data on an hypersurface $\mathcal{S}$ can be chosen such that

$$S_{ab} = 0, \quad \nabla_c S_{ab} = 0, \qquad \text{on} \;\; \mathcal{S}$$

then because of the homogeneity of the wave equation for S_{ab}, it follows that necessarily $S_{ab} = 0$ in the development of $\mathcal{S}$ so that ξ^a is, in fact, a Killing vector. The conditions are called the **Killing initial Data (KID) conditions**.

Intrinsic conditions (I)

A $3+1$ split:

The KID equations are conditions not only on ξ^a but also on the initia data $(\mathcal{S}, h_{ij}, K_{ij})$. Writing

$$\xi^a = Nn^a + N^a, \qquad n_a N^a = 0,$$

where N and N^a denote the **lapse and shift of the Killing vector**, A computation then shows that the space-space components of the Killing equation $\nabla_a \xi_b + \nabla_b \xi_a = 0$ imply

$$NK_{ij} + D_{(i}Y_{j)} = 0.$$

Taking a time derivative of the above equation and using the ADM evolution equations one finds that

$$N^k D_k K_{ij} + D_i N^k K_{kj} + D_j N^k K_{ik} + D_i D_j N = N(r_{ij} + KK_{ij} - 2K_{ik}K^k{}_j).$$

Intrinsic conditions (II)

Theorem

Let $(\mathcal{S}, h_{ij}, K_{ij})$ *denote an initial data set for the vacuum Einstein field equations. If there exists a pair* (N, N^i) *such that*

$$NK_{ij} + D_{(i}Y_{j)} = 0,$$
$$N^k D_k K_{ij} + D_i N^k K_{kj} + D_j N^k K_{ik} + D_i D_j N = N(r_{ij} + KK_{ij} - 2K_{ik}K^k{}_j),$$

then the development of the initial data has a Killing vector.

Remarks:

- The KID conditions are overdetermined. This is natural as not every spacetime admits a symmetry.
- The KID conditions are closely related to the constraint equations and the ADM evolution equations. This is a deep relation which will not be explored here!

Outline

Where are we?

The Cauchy problem for the Einstein Equations

- As already pointed out, General Relativity satisfies the remarkable fact that it admits a formulation in terms of an initial value problem.
- The original formulation and proof of this statement is due to Y. Choquet-Bruhat (1952).
- A satisfactory formulation which provides geometric uniqueness is due to Y. Choquet-Bruhat (1968).

A first version of the theorem

Theorem

Given a solution to the Einstein constraint equations of a 3-dimensional manifold $\mathcal{S}$*, there exists an* $\varepsilon > 0$ *such that on* $\mathcal{S} \times [0, \varepsilon)$ *there is a metric* g_{ab} *solving the Einstein field equations. The metric* g_{ab} *implies the provided solution to the constraints on* $\mathcal{S}$*,*

Remark:

- The pair $(\mathcal{S} \times [0, \varepsilon), g_{ab})$ is called a **hyperbolic development** of the solution to the constraint equations.
- A priori there is no controll on the size of ε unless one provides more information.

On the proof of the theorem

Structure of the proof:

- Make use of the wave coordinates condition to obtain the reduced Einstein equations of the form

$$g^{\lambda\rho}\partial_\lambda\partial_\rho g_{\mu\nu} + H_{\mu\nu}(g,\partial g) = 0.$$

- The general theory of systems of quasilinear wave equations ensure the existence of solutions for a short interval of time ε is suitable initial data $g^\star_{\mu\nu}$ and $(\partial_\lambda g_{\mu\nu})^\star$ is provided on $\mathcal{S}$.
- The initial data is built from the solution to the constraint equations. In particular, the 3-metric $h_{\alpha\beta}$ provides the spatial part of $g_{\mu\nu}$. The components g_{00} and $g_{0\alpha}$ are obtained by the prescription of lapse (α) and shift (β^α) implied by the wave coordinates.

The reduced Einstein equations vs the Einstein equations

Some remarks:

- At the begining of the lectures it was shown that if the coordinates satisfy the wave coordinate condition $\Box x^\mu = 0$, then the Einstein field equations reduce to a system of quasilinear wave equations for the components of the metric $g_{\mu\nu}$ —the so-called **reduced Einstein equations**.
- To conclude the discussion it is now shown that under suitable conditions the Eistein reduced equations imply a solution of the actual Einstein field equations.
- This in fact, is equivalent to showing that if the contracted Christoffel symbols $\Gamma^\mu \equiv g^{\nu\lambda}\Gamma^\mu{}_{\nu\lambda}$ vanish initially, then they also vanish at any later time.

Propagating the wave coordinate condition

A computation:

The key observation is that the *reduced Einstein field equations* can be written as

$$R_{\mu\nu} = \nabla_{(\mu}\Gamma_{\nu)}.$$

Now, using the contracted Bianchi identity

$$\nabla^{\mu}(R_{\mu\nu} - \tfrac{1}{2}Rg_{\mu}) = 0,$$

it follows that

$$\Box\Gamma_{\mu} + R^{\nu}{}_{\mu}Q_{\mu} = 0.$$

This is a wave equation for the contracted Christoffel symbol. In view of its homogeneity, if

$$\Gamma_{\mu} = 0, \qquad \nabla_{\nu}\Gamma_{\mu} = 0, \qquad \text{on } \mathcal{S}$$

then $\Gamma_{\mu} = 0$ in the development of $\mathcal{S}$ and accordingly $R_{\mu\nu} = 0$.

Remark:

The initial conditions on the contracted Christoffel symbols are related to the constraint equations.

Issues with uniqueness

Remark:

- The hyperbolic development depends on the choice of α and β^α.
- Different choices of lapse and shift give rise to different hyperbolic developments $(\mathcal{S} \times [0, \varepsilon_1))$ and $(\mathcal{S} \times [0, \varepsilon_2))$.
- On the intersection of the developments the metrics are related by a coordinate transformation.
- However, there is, in principle, an infinite number of hyperbolic developmemts.

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Question:

Can one find a satisfactory formulation of the issue of uniqueness?

Addressing geometric uniqueness

Maximal hyperbolic development:

A **maximal hyperbolic development** of a solution to the constraints is a hyperbolic development which contains any other.

Addressing geometric uniqueness

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Theorem (Y. Choquet-Bruhat & R. Geroch, 1968)

Given a solution to the constraints on $\mathcal{S}$ there exists a unique maximal hyperbolic development $(\mathcal{M}_\bullet, g^\bullet_{ab})$.

Remarks:

- The set of hyperbolic developments is endowed with a partial order structure.
- The proof of the theorem makes use of **Zorn's lemma**: every bounded set endowed with a partial order has a maximal element.
- **BIG QUESTION:** how to obtain the maximal hyperbolic development of a solution to the constraints?