Mathematical problems of General Relativity
Problem sheet 3

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1. Show that in the non-vacuum case

$$R_{ab} - \tfrac{1}{2}Rg_{ab} = T_{ab},$$

the constraint equations are given by

$$r + K^2 - K_{ab}K^{ab} = \rho,$$
$$D^bK_{ab} - D_aK = J_a,$$

where

$$\rho \equiv T_{ab}n^a n^b, \qquad J_a \equiv -h_a{}^b n^c T_{bc}$$

denote, respectively, the energy density and momentum density vector with respect to the normal observer n^a.

2. Show that the Lie derivative of the projector $h_a{}^b$ along αn^a vanishes —i.e.

$$\mathcal{L}_{\alpha n} h_a{}^b = 0.$$

3. Show that the Lie derivative of any spatial tensor along αn^a is again spatial.

4. Show that the determinant $g = \det(g_{\mu\nu})$ of the components of the spacetime metric can be written as

$$g = -\alpha^2\gamma,$$

where $\gamma = \det(\gamma_{ij})$. Hint: recall the formula for the inverse of a square matrix.

5. Show that the determinant γ of the spatial metric and the trace $K = K_i{}^i$ of the extrinsic curvature satisfy the equations

$$\partial_t \ln \gamma^{1/2} = -\alpha K + D_i\beta^i,$$
$$\partial_t K = -\gamma^{ij}D_iD_j\alpha + \alpha K_{ij}K^{ij} + \beta^i D_i K.$$

6. Show that for any spatial vector V^i one has that

$$\nabla_i V_j = D_i V_j,$$

but that, in general,

$$\nabla_i V^j \neq D_i V^j, \qquad \nabla^i V_j \neq D^i V_j.$$

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7. In spherical *Painlevé-Gullstrand* coordinates, the line element of the Schwarzschild spacetime is given by

$$g = -\mathrm{d}t^2 + \left(\sqrt{\frac{2m}{r}}\mathrm{d}t + \mathrm{d}r\right)^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\varphi^2).$$

Identify the lapse, shift and the 3-metric for this form of the spacetime metric with respect to the foliation given by hypersurfaces of constant t —note: the 3-metric should be flat! Compute the extrinsic curvature.