Mathematical problems of General Relativity
Lecture 3

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Outline

1. The $3+1$ decomposition of General Relativity
   - The 3+1 form of the spacetime metric
2. A closer look at the constraint equations
3. Time independent solutions

Outline

Adapted coordinates (I)

Remarks:

- The discussion of the evolution equations given in the previous section has been completely general.
- The only assumption that has been made about the spacetime is that it is globally hyperbolic so that a foliation and a corresponding time vector exist.
- The discussion of the $3+1$ can be further particularised by introducing adapted coordinates. In this section we briefly discuss how this can be done.

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Choosing the time coordinate:

- Recall that the hypersurfaces of the foliation of a spacetime $(\mathcal{M}, g_{ab})$ can be given as the level surfaces of a time function t.
- We already have seen that $\nabla_a t^a = 1$. The latter combined with $\nabla_\mu t = (1,0,0,0)$ readily imply that $t^\mu = (1,0,0,0)$. Hence, the Lie derivative along the direction of t^a is simply a partial derivative —that is,

$$\mathcal{L}_t = \partial_t.$$

Adapted coordinates (II)

The shift vector:

- From the previous discussion it follows that the spatial components of the unit normal must vanish —i.e. $n_\alpha = 0$.
- Since the contraction of spatial vectors with the normal must vanish, it follows that all components of spatial tensors with a contravariant index equal to zero must vanish.
- For the shift vector one has that $n_\mu \beta^\mu = n_0 \beta^0 = 0$ so that $\beta^\mu = (0, \beta^\alpha)$.
- Since one has that $t^a = \alpha n^a + \beta^a$, it follows then that

$$n^\mu = (\alpha^{-1}, -\alpha^{-1}\beta^\alpha).$$

- Moreover, from the normalisation condition $n_a n^a = -1$ one finds

$$n_\mu = (-\alpha, 0, 0, 0).$$

Adapted coordinates (III)

The 3-metric:

- Recalling that $h_{ab} = g_{ab} + n_a n_b$ one concludes that

$$h_{\alpha\beta} = g_{\alpha\beta}.$$

- In these **adapted coordinates** the 3-metrics of the hypersurfaces of the foliation are simply the spatial part of the spacetime metric g_{ab}.
- Moreover, since the time components of spatial contravariant tensors have to vanish, one also has that $h^{\mu 0} = 0$.
- One concludes that one can write

$$g^{\mu\nu} = h^{\mu\nu} - n^\mu n^\nu = \begin{pmatrix} -\alpha^{-2} & \alpha^{-2}\beta^\gamma \\ \alpha^{-2}\beta^\delta & h^{\gamma\delta} - \alpha^{-2}\beta^\gamma\beta^\delta \end{pmatrix}.$$

- This last expression can be inverted to yield

$$g_{\mu\nu} = \begin{pmatrix} -\alpha^2 + \beta_\gamma\beta^\gamma & \beta_\gamma \\ \beta_\gamma & h_{\gamma\delta} \end{pmatrix},$$

where $\beta_\gamma \equiv h_{\gamma\delta}\beta^\delta$.

The 3+1 form of the metric

The line element in adapted coordinates:

- An alternative way of presenting the latter is via the line element

$$g = -\alpha^2 \mathrm{d}t^2 + h_{\gamma\delta}(\beta^\gamma \mathrm{d}t + \mathrm{d}x^\gamma)(\beta^\delta \mathrm{d}t + \mathrm{d}x^\delta).$$

- The latter is known as the $3+1$ **form** of the spacetime metric.

The constraint and evolution equations in adapted coordinates

A summary:

- The constraint and ADM evolution equations can be written in adapted coordinates as

$$
\begin{aligned}
&r + K^2 - K_{ij}K^{ij} = 0, \\
&D^j K_{ij} - D_i K = 0, \\
&\partial_t h_{ij} = -2\alpha K_{ij} + D_i\beta_j + D_j\beta_i, \\
&\partial_t K_{ij} = -D_i D_j \alpha + \alpha(r_{ij} - 2K_{ik}K^k{}_j + K K_{ij}) \\
&\qquad\qquad + \beta^k D_k K_{ij} + K_{ik} D_j \beta^k + K_{kj} D_i \beta^k.
\end{aligned}
$$

An example: the Schwarzschild spacetime (I)

Isotropic coordinates:

- The metric Schwarzschild spacetime can be expressed in standard coordinates in terms of the line element

$$g = -\left(1 - \frac{2m}{r}\right)\mathrm{d}t^2 + \left(1 - \frac{2m}{r}\right)^{-1}\mathrm{d}r^2 + r^2\mathrm{d}\theta^2 + r^2\sin^2\theta\mathrm{d}\varphi^2.$$

- This form of the metric is not the best one for a $3+1$ decomposition of the spacetime.
- Instead, introduce an **isotropic radial coordinate** $\bar{r}$ via $r = \bar{r}\left(1 + \frac{m}{2\bar{r}}\right)^2$.
- In terms of the later one obtains the line element of the Schwarzschild spacetime in the form

$$g = -\left(\frac{1 - m/2\bar{r}}{1 + m/2\bar{r}}\right)^2 + \left(1 + \frac{m}{2\bar{r}}\right)^4(\mathrm{d}\bar{r}^2 + \bar{r}^2\mathrm{d}\theta^2 + \bar{r}^2\sin^2\theta\mathrm{d}\varphi).$$

An example: the Schwarzschild spacetime (II)

The gauge functions:

- The normal $\omega_a = \nabla_a t$ is then readily given by

$$\omega_\mu = (1,0,0,0).$$

- Thus, one readily reads the lapse function to be

$$\alpha = \frac{1 - m/2\bar{r}}{1 + m/2\bar{r}},$$

  while the unit normal is

$$n^\mu = \frac{1 + m/2\bar{r}}{1 - m/2\bar{r}}(1,0,0,0).$$

- Also, the shift vanishes: $\beta^\alpha = 0$.

An example: the Schwarzschild spacetime (III)

The intrinsic metric and the extrinsic curvature:

- The spatial metric is then

$$h = \left(1 + \frac{m}{2\bar{r}}\right)^4 (\mathrm{d}\bar{r}^2 + \bar{r}^2 \mathrm{d}\theta^2 + \bar{r}^2 \sin^2\theta \mathrm{d}\varphi).$$

- Since $\beta^i = 0$ and h_{ij} is independent of time, one can readily finds that the extrinsic curvature vanishes

$$K_{ij} = 0.$$

- The isotropic form of the Schwarzschild readily yields a foliation of spacetime that follows the static symmetry of the spacetime.
- In this foliation, the intrinsic 3-metric of the leaves does not seems to evolve.
- Any other foliation not aligned with the static Killing vectors will give rise to a non-trivial evolution for both h_{ij} and K_{ij}.

Outline

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    - The 3+1 form of the spacetime metric

2. A closer look at the constraint equations

3. Time independent solutions

The constraint equations —summary (I)

The constraint equations:

- The Einstein field equations imply the following constraint equations on a (spatial) hypersurface $\mathcal{S}$:

$$r + K^2 - K_{ij}K^{ij} = 0, \qquad \text{Hamiltonian constraint}$$
$$D^i K_{ij} - D_j K = 0. \qquad \text{Momentum constraint}$$

- These equations constraint the possible choices of pairs (h_{ij}, K_{ij}) corresponding to initial data to the Einstein field equations.
- They are intrinsic equations, that is, they only involve objects which are defined on the hypersurface $\mathcal{S}$ without any further reference to the "bulk" of the spacetime $(\mathcal{M}, g_{ab})$.

The constraint equations —summary (II)

PDE properties:

- The Einstein constraint represent a highly coupled, highly non-linear system of equations for (h_{ij}, K_{ij}).
- The main difficulty in constructing an solution to the equations lies in the fact that the equations constitute an underdetermined system: one has 4 equations for 12 unknowns —the independent components of two symmetric spatial tensors.
- Even exploiting the coordinate freedom to "kill off" three components of the tensors, one is still left with 9 unknowns.
- There should be some freedom in the specification of **data** for the equations.
  - The task is to identify what this free data is!!!

Simplifying assumptions: time symmetry

Vanishing of the extrinsic curvature

- In order to render the problem manageable, we make a standard simplifying assumption and consider initial data sets for which $K_{ij} = 0$ everywhere on $\mathcal{S}$.
- This class of initial data are called **time symmetric**.
- The reason for this is that if $K_{ij} = 0$ at $\mathcal{S}$ then the evolution equations imply that

$$\partial_t h_{ij} = 0, \qquad \text{on} \qquad \mathcal{S}.$$

  This equation is invariant under the replacement $t \mapsto -t$.
- It follows that the resulting spacetime has a reflection symmetry with respect to the hypersurface $\mathcal{S}$ which can be regarded as a **moment of time symmetry**.

The time symmetric constraints

The Hamiltonian constraint:

- If $K_{ij} = 0$ everywhere on $\mathcal{S}$ then the momentum constraint is automatically solved, and the Hamiltonian constraint reduces to

$$r = 0.$$

- That is, the initial 3-metric has to be such that its Ricci scalar —notice that this does not mean that the hypersurface is flat!
- The time symmetric Hamiltonian constraint regarded as an equation for h_{ij} is highly non-linear.
- Moreover, one still has 6 unknowns and equation —even choosing coordinates, one still has 3 unknowns .

Conformal rescalings and the Yamabe problem

A strategy:

- Clearly, for an arbitrary metric $\bar{h}_{ij}$ one has that $\bar{r} \neq 0$.
- An idea to solve the constraint is to introduce a factor that compensates this.
- This idea leads naturally to the notion of conformal transformations.

Conformal rescalings and the Yamabe problem

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Conformal rescalings:

- Two metrics h_{ij}, $\bar{h}_{ij}$ are said to be **conformally related** if there exists a positive scalar ϑ (the **conformal factor**) such that

$$h_{ij} = \vartheta^4 \bar{h}_{ij}.$$

- The metric $\bar{h}$ will be called the **background** metric.
- Loosely speaking, the conformal factor absorbs the overall scale of the metric.
- At the level presented here, the conformal transformation introduced above is just a mathematical trick to solve equations. At a deeper level, the conformal transformation defines an equivalence class of manifolds and metrics.

More on conformal transformations

Transformation laws for derived objects:

- The 3-dimensional Christoffel symbols are given by

$$\Gamma^{\alpha}{}_{\beta\gamma} = \tfrac{1}{2} h^{\alpha\delta}(\partial_\beta h_{\gamma\delta} + \partial_\gamma h_{\beta\delta} - \partial_\delta h_{\beta\gamma}).$$

- It follows that

$$\Gamma^{\alpha}{}_{\beta\gamma} = \bar{\Gamma}^{\alpha}{}_{\beta\gamma} + 2(\delta_\beta{}^\alpha \partial_\gamma \ln\vartheta + \delta_\gamma{}^\alpha \partial_\beta \ln\vartheta - \bar{h}_{\beta\gamma}\bar{h}^{\alpha\delta}\partial_\delta \ln\vartheta).$$

- A lengthier computation yields the following transformation law for the Ricci tensor:

$$\begin{aligned} r_{ij} = \bar{r}_{ij} - 2(&\bar{D}_i\bar{D}_j \ln\vartheta + \bar{h}_{ij}\bar{h}^{lm}\bar{D}_l\bar{D}_m \ln\vartheta) \\ &+4(\bar{D}_i \ln\vartheta \bar{D}_j \ln\vartheta - \bar{h}_{ij}\bar{h}^{lm}\bar{D}_l \ln\vartheta \bar{D}_m \ln\vartheta). \end{aligned}$$

- Furthermore (and more importantly for our purposes) one has that

$$r = \vartheta^{-4}\bar{r} - 8\bar{\theta}^{-5}\bar{D}_k\bar{D}^k\vartheta.$$

In the above expressions $\bar{D}$ denotes the covariant derivative of the background metric $\bar{h}_{ij}$.

The Yamabe equation (I)

The Hamiltonian constraint and conformal rescalings:

- Using $r = 0$ in the transformation law for the Ricci scalar given above, one readily finds that
$$\bar{D}_k \bar{D}^k \vartheta - \tfrac{1}{8}\bar{r}\vartheta = 0.$$
- This equation is sometimes called the **Yamabe equation** in Differential Geometry.
- Given a fixed background metric $\bar{h}_{ij}$, then it can be read as an equation for the conformal factor ϑ.
- Given a solution ϑ, one has that by construction $h_{ij} = \vartheta^4 \bar{h}_j$ is such that $r = 0$ and one has constructed a solution to the time symmetric Einstein constraints.

The Yamabe equation (II)

Mathematical properties of the Yamabe equation

- The Yamabe equation is **elliptic**:
  - the operator $\bar{D}_k \bar{D}^k$ is the **Laplacian operator** associated to the metric $\bar{h}_{ij}$;
  - if $\bar{h}_{\alpha\beta} = \delta_{\alpha\beta}$ the flat metric in Cartesian coordinates, then

$$\bar{D}_k \bar{D}^k = \delta^{\alpha\beta}\partial_\alpha \partial_\beta = \partial_x^2 + \partial_y^2 + \partial_z^2.$$

- Given a linear second order elliptic equation appropriate **boundary conditions** ensure the existence of a unique solution on $\mathcal{S}$.

A further simplifying assumption: conformal flatness

What is conformal flatness?

- Choose the flat metric as background metric. That is, let

$$\bar{h}_{\alpha\beta} = \delta_{\alpha\beta}.$$

- In this case, the metric $h_{\alpha\beta} = \vartheta^4 \delta_{\alpha\beta}$ is said to be **conformally flat**.
- Conformal flatness is an interesting property that Riemannian manifolds can possess. An important result is that conformal flatness is characterised locally by the vanishing of the **Cotton-York** tensor

$$b_{ijk} \equiv D_{[j} r_{k]i} - \tfrac{1}{4} h_{i[j} D_{k]} r.$$

- For example, any spherically symmetric metric can be shown to be conformally flat.
- Conformal flatness simplifies the calculations that need to be carried out.
- One has that $\bar{r} = 0$ so that the Yamabe equation reduces to the **flat Laplace equation**

$$\bar{D}_k \bar{D}^k \vartheta = 0.$$

Boundary conditions

Describing isolated systems:

- In the discussion of isolated systems (i.e. astrophysical sources) one is interested in solutions which are **asymptotically flat**. That is,

$$\vartheta = 1 + O(r^{-1}), \qquad \text{for} \qquad r \to \infty,$$

where $r^2 = x^2 + y^2 + z^2$ is the standard radial coordinate.

- Solutions to the Laplace equation with the above asymptotic behaviour are well known. In particular, a **spherically symmetric** solution is given by

$$\vartheta = 1 + \frac{m}{2r},$$

where m is a constant.

Data for the Schwarzschild spacetime (I)

Interpreting the solution:

- Given
$$\vartheta = 1 + \frac{m}{2r},$$
the associated solution to the Hamiltonian constraint is the 3-metric of the Schwarzschild spacetime in isotropic coordinates:
$$h = \left(1 + \frac{m}{2r}\right)^4 (\mathrm{d}r^2 + r^2\mathrm{d}\theta^2 + r^2 \sin^2\theta \mathrm{d}\varphi^2).$$
- The above 3-metric is singular at $r = 0$. This singularity, is a coordinate singularity. By considering the coordinate inversion
$$r = \frac{m^2}{4}\frac{1}{\bar{r}},$$
the metric transforms into
$$h = \left(1 + \frac{m}{2\bar{r}}\right)^4 (\mathrm{d}\bar{r}^2 + \bar{r}^2\mathrm{d}\theta^2 + \bar{r}^2 \sin^2\theta \mathrm{d}\varphi^2).$$
- The inversion transforms the metric into itself —that is, it is a discrete isometry. In particular, one has that the point $r = 0$ is can be mapped to infinity. Thus, the metric is perfectly regular everywhere and $r = 0$ is, in fact, the infinity of an asymptotically flat region.

Data for the Schwarzschild spacetime (II)

The Einstein-Rosen bridge:

- The hypersurface $\mathcal{S}$ has a non-trivial topology —it corresponds to a *wormhole*.
- The radius given by $r = m/2$ corresponds to the minimum areal radius —this is called the *throat* of the black hole.
- The throat corresponds to the intersection of the black hole horizon with the hypersurface $\mathcal{S}$. The inversion reflects points with respect to the throat.

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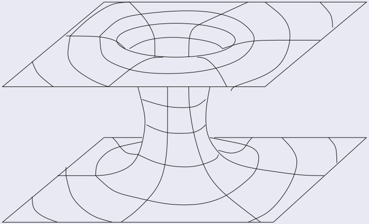
Embdedding diagram of the Schwarzschild data:



Multiple black hole data (I)

Brill-Lindquist initial data:

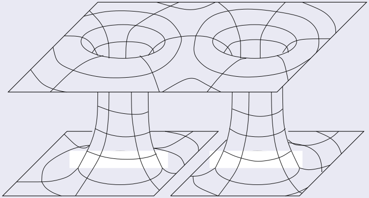
- The construction described in the previous paragraphs can be extended to include an arbitrary number of black holes.
- This is made possible by the linearity of the flat Laplace equation.
- Indeed, the conformal factor

$$\vartheta = 1 + \frac{m_1}{2r_1} + \frac{m_2}{2r_2}, \qquad r_1 = |x^i - x_1^i|, \qquad r_2 = |x^{-}x_2^i|,$$

  where x_1^i and x_2^i denote the (fixed) location of two black holes with **bare masses** m_1 and m_2.
- The solution is called the **Brill-Lindquist solution**.
- It describes a pair of black holes instantaneosly at rest at a moment of time symmetry. This solution is much used as initial data to simulate the head-on collision of two black holes.
- One finds is that each throath connects to is own asymptotically flat region. The drawing of the corresponding 3-dimensional manifold gives 3 different sheets, each corresponding to a different asymptotically flat region.

Multiple black hole data (II)

Embedding diagram of the Brill-Lindquist data:



Multiple black hole data (III)

The Misner initial data:

- The flat Laplace equation can also be solved using the so-called **method of images** to obtain a solution with two holes and two asymptotic regions.
- This solution is known as **Misner data**.
- This solution has a reflection symmetry through the throaths, and has only two (as oppossed to three of the Brill-Lindquist solution) asymptotically flat regions.
- The solution can be also interpreted as a worm hole data by making suitable topological identifications.

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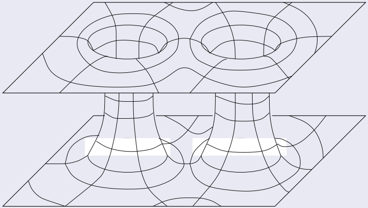

Concluding remarks concerning the constraint equations

Remarks:

- More complicated solutions to the constraint equations can be obtained by including a non-vanishing extrinsic curvature.
- In this way one can provide data for a rotating black hole or even a pair of rotating black holes spiralling around each other.
- The constraint equations in these cases have to be solved numerically.

Outline

Time independent solutions

Motivation:

- A systematic analysis of solutions to the vacuum Einstein field equations must start by considering time independent solutions.
- These solutions are interpreted as describing the gravitational field in the exterior of **isolated bodies at rest** or in **uniform rotation** in an otherwise empty Universe.
- The simplest case of a time independent solution is given by the Minkowski metric.
- More sophisticated examples are given by the **Schwarzschild** and **Kerr spacetimes**.
  - The relevance of these two solutions is that they are thought to describe, in a suitable sense, the **end state of black hole evolution**.

A toy model: the wave equation (I)

Time independent solutions to the wave equation

- Consider a scalar field on Minkowski spacetime satisfying the wave equation

$$(\Delta - \partial_t^2)\phi = 0,$$

  where Δ denote the flat Laplacian.
- For time independent solutions —i.e. $\partial_t \phi = 0$— it follows that

$$\Delta \phi = 0.$$

- An equation which is origially **hyperbolic** becomes **elliptic** under the assumption of time independence.
  - This is a generic feature that can be observed in other theories —like the Maxwell equations and the Einstein field equations.

A toy model: the wave equation (II)

Boundary conditions:

- The energy of the scalar field at some time t is given by

$$E(t) = \int_{\mathcal{S}_t} \left((\partial_t \phi)^2 + |\nabla \phi|^2 \right) \mathrm{d}^3 x.$$

- In order to have finiteness of the energy one needs the boundary conditions

$$\phi(t, x^i), \; \partial_t \phi(t, x^i) \to \infty, \qquad \text{as} \qquad |x| \to \infty.$$

A toy model: the wave equation (II)

Ellipticity properties:

- An important difference between hyperbolic equations and elliptic ones is that while in the former, properties of solutions can be localised and have finite propagation speed, **for the latter the properties of solutions are global**.
- For example, if $\phi = O(1/r)$ as $r \to \infty$ and $\Delta\phi = 0$, then it follows that $\phi = 0$.
- This follows from

$$0 = \int_{\mathbb{R}^3} \phi\Delta\phi \mathrm{d}x^3 = \int_{\mathbb{R}^3} |\nabla\phi|^2 \mathrm{d}x^3,$$

  where Green's identity has been used. It follows that $|\Delta|^2 = 0$ everywhere on $\mathbb{R}^3$ so that ϕ is constant.
- Due to the decay conditions, it must necessarily vanish.
- This type of argument will be used repeatedly for the Einstein equations.
- In order to avoid the vanishing of ϕ in this case, one needs to consider the inhomogeneous problem —that is, one needs to consider sources.

Stationarity and static solutions in GR (I)

Intuition:

- Mathematically speaking, time independence is imposed by requiring on the spacetime $(\mathcal{M}, g_{ab})$ the existence of timelike KIlling vector ξ^a —the spacetime is then said to be **stationary**.
- If, in addition, the Killing vector is hypersurface orthogonal —i.e. it is the gradient of some scalar function— then one says that ξ^μ is **static**.
- The Schwarzschild and Kerr solutions are, respectively, static and stationary.
- **Stationary solutions** to the Einstein field equations allow for the possibility of **rotating gravitational fields**.

Stationarity and static solutions in GR (II)

Frobenius theorem:

- Let n_a denote the unit normal of an hypersurface $\mathcal{S}$.
- If $\xi^a n_a$, i.e. the Killing vector if orthogonal to $\mathcal{S}$, then a calculation readily shows that
$$\xi_{[a}\nabla_b\xi_{c]} = 0.$$
- The latter condion characterises **hyperusrface orthogonality** —that is, a Killing vector is hypersurface orthogonal if and only if the previous equation holds

The static vacuum equations (I)

The metric of a static spacetime:

- In a stationary spacetime, it is natural to choose adapted coordinates such that $\xi^\mu \partial_\mu = \partial_t$ —that is, the time coordinate is adapted to the flow lines of the Killing vector.
- Using the Killing vector condition $\mathcal{L}_\xi g_{ab} = 0$ and the definitions of h_{ij} and K_{ij} one can show that
$$\partial_t h_{ij} = \partial_t K_{ij} = 0.$$
- If the Killing vector is hypersurface orthogonal then it follows that the Killing vector has to be proportional to the normal to the hypersurface $\mathcal{S}$:
$$\xi_\mu = \alpha \nabla_\mu t$$
- However, the Killing vector can be decomposed in a lapse and a shift part:
$$\xi^a = N n^a + \beta^a.$$
- Comparing both expressions one necessarily has that $\beta^\alpha = 0$.
- Thus, one has that
$$g = -\alpha^2 \mathbf{d}t^2 + h_{\alpha\beta} \mathbf{d}x^\alpha \mathbf{d}x^\beta,$$
with $h_{\alpha\beta}$ time independent.

The static vacuum equations (II)

Vanishing of the extrinsic curvature:

- The time evolution equation for h_{ij} then takes the form

$$\partial_t h_{ij} = -2\alpha K_{ij} = 0.$$

- As the lapse cannot vanish one has that

$$K_{ij} = 0.$$

That is, the hypersurfaces of the foliation adapted to the static Killing vector have no extrinsic curvature —this property is preserved as $\partial_t K_{ij} = 0$.

The static vacuum equations (III)

The equations:

- Vacuum static solutions are characterised solely in terms of the lapse α and the 3-metric h_{ij}.
- In order to obtain equations for these quantities one considers the Hamilitonian constraint and the evolution equation for K_{ij}.
- Setting $K_{ij} = \partial_t K_{ij} = 0$ readily yields

$$D_i D_j \alpha = r_{ij},$$
$$r = 0,$$

where, as before, r denotes the Ricci tensor of the 3-metric h_{ij}. These equations are known as the **static vacuum Einstein equations**.

Properties of the static equations: Licnerowicz theorem (I)

Assumptions and boundary conditions:

- As a first example of the content and implications of the static equations let $\mathcal{S} \approx \mathbb{R}^3$
  - i.e. the hypersurfcae has the topology of Euclidean space.
- Suppose that the fields α and $h_{\alpha\beta}$ decay at infinity in such a way that

$$\alpha \to 1, \qquad h_{\alpha\beta} - \delta_{\alpha\beta} \to 0, \qquad \text{as} \qquad r \to \infty.$$

- The first condition essentially means that it is assumed that the Killing vector behaves asymptotically like the static Killing vector of Minkowski spacetime.
- The second condition means that the 3-metic is assumed to be asymptotically flat (Euclidean) at infinity.

Properties of the static equations: Licnerowicz theorem (II)

Ellipticity and the static equations:

- Taking traces of the first static equation and using the second equation it follows that
$$\Delta\alpha = D_k D^k \alpha = 0.$$
- Now, consider
$$0 = \int_{\mathcal{S}} \alpha \Delta \alpha \mathrm{d}^3 x = \int_{\mathcal{S}} |D\alpha|^2 \mathrm{d}^3 x,$$
again, as a consequence of Green's identity.
- Thus
$$|D\alpha|^2 = h^{ij} D_i \alpha D_j \alpha = 0,$$
from where it follows that α is a constant.
- Using the asymptotic condition $\alpha \to 1$ it follows $\alpha = 1$ everywhere.

Properties of the static equations: Licnerowicz theorem (III)

Flatness of the 3-metric:

- Using the first static equation one concludes that

$$r_{ij} = 0.$$

- In 3-dimensions the Ricci tensor determines fully the curvature of the manifold. Thus

$$r_{ijkl} = 0.$$

  That is, $h_{\alpha\beta} = \delta_{\alpha\beta}$ —the Euclidean flat metric.

- The solution we have obtained then is

$$g = -\mathrm{d}t^2 + \delta_{\alpha\beta}\mathrm{d}x^\alpha \mathrm{d}x^\beta.$$

  This solution is the Minkowski spacetime! This result is known as **Licnerowicz's theorem**.

Properties of the static equations: Licnerowicz theorem (IV)

Theorem

The only globally regular static solution to the Einstein equations with $\mathcal{S}$ having trivial topology (i.e. $\mathcal{S} \approx \mathbb{R}^3$) and such that

$$\alpha \to 1, \qquad h_{\alpha\beta} - \delta_{\alpha\beta} \to 0, \qquad \text{as} \qquad r \to \infty$$

is the Minkowski spacetime.

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is the Minkowski spacetime.

Morally:

- The above theorem demonstrates the rigidity of the Einstein field equations.
- In order to obtain more interesting regular solutions, one requires either some matter sources or a non-trivial topology for $\mathcal{S}$ as in the case of the Schwarzschild spacetime **—recall the Einstein-Rosen bridge!**
- The result can be interpreted as a first, very basic **uniqueness black hole result**.
- If one wants to have a black hole solution one needs non-trivial topology!

Further properties of static solutions: leading asymptotic behaviour

Question:

- An important question when analysing static spacetimes is to analyse their asymptotic behaviour beyond the prescribed boundary conditions.
- Can one say more?

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Question:

- An important question when analysing static spacetimes is to analyse their asymptotic behaviour beyond the prescribed boundary conditions.
- Can one say more?

Theorem (Beig,1980)

Every static vacuum solution to the Einstein equations satisfying

$$\alpha \to 1, \qquad h_{\alpha\beta} - \delta_{\alpha\beta} \to 0, \qquad \textit{as} \qquad r \to \infty$$

is Schwarzschildean to leading order in $1/r$*. That is,*

$$\alpha^2 = 1 - \frac{2m}{r} + O(1/r^2), \qquad h_{\alpha\beta} - \delta_{\alpha\beta} = \frac{2m}{r}\delta_{\alpha\beta} + O(1/r^2).$$

Remarks:

- Notice that in the previous result the regularity of $\mathcal{S}$ is not required. Also, there could be bounded sources somewhere in the interior.

Further properties of static solutions: multipole moments

Defining multipole moments:

- The lapse α can be interpreted as relativistic generalisation of a **Newtonian potential**.
- The previous theorem on the leading behaviour of static solutions theorem can be improved to include **higher order multipoles**.
- These lead to a *multipolar expansion* of the gravitational field.
- These mutipoles characterise in a unique manner static solutions.

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Theorem (Beig & Simon, 1981; Friedrich 2006)

Given an asymptotically flat static solution to the Einstein vacuum equations, one obtains a unique sequence of multipole moments. Conversely, given a sequence of multipole moments, if the lapse constructed from this sequence exists, there exists a unique static spacetime associated to these multipoles.