Mathematical problems of General Relativity
Problem sheet 2

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1. Show that the unit normal n^a is rotation free. That is, one has that
$$n_{[a}\nabla_b n_{c]} = 0.$$
Moreover show that the *twist* $\omega_{ab} \equiv h_a{}^c h_b{}^d \nabla_{[c} n_{d]}$ vanishes.

2. Given an arbitrary spacetime vector v^a show that $h_a{}^b v^a$ with $h_{ab} = g_{ab} + n_a n_b$ is purely spatial.

3. Given an arbitrary tensor T_{ab} show that one can write
$$T_{ab} = T^{\perp}_{ab} - n_a n^c T^{\perp}_{cb} - n_b n^c T^{\perp}_{ac} + n_a n_b n^c n^d T_{cd}.$$

4. Given the Schwarzschild metric in isotropic coordinates
$$g = -\left(\frac{1-\frac{m}{2r}}{1+\frac{m}{2r}}\right)^2 \mathrm{d}t^2 + \left(1+\frac{m}{2r}\right)^2 (\mathrm{d}r^2 + r^2\mathrm{d}\theta^2 + r^2\sin^2\theta \mathrm{d}\varphi^2),$$
consider the foliation given by hypersurfaces with constant time coordinate t. Compute the covector ω_a, the lapse of the foliation, its unit normal and the spatial metric of the hypersurfaces. Show that the extrinsic curvature of the hypersurfaces vanishes.

5. Show that the 3-dimensional covariant derivative D_a is compatible with the spatial metric h_{ab}. Moreover, show that D_a is torsion free.

6. Show that for the scalar product $v^a\omega_a$, the Leibnitz rule
$$D_a(v^b\omega_b) = v^b D_a\omega_b + \omega_b D_a v^b$$
holds only if v^a and ω_a are purely spatial.

7. Show that the acceleration is purely spatial.

8. Show that the acceleration a_a is related to the lapse α according to
$$a_a = D_a \ln \alpha.$$

9. Show that the acceleration for the normal observer in the Schwarzschild spacetime vanishes.

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