Mathematical problems of General Relativity
Lecture 2

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Outline

1. The $3+1$ decomposition of General Relativity
   - Submanifolds of spacetime
   - Foliations of spacetime
   - The intrinsic metric of an hypersurface
   - The extrinsic curvature of an hypersurface
   - The Gauss-Codazzi and Codazzi-Mainardi equations
   - The constraint equations of General Relativity
   - The ADM-evolution equations

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Submanifolds

Intuitive definition:

- A **submanifold** of $\mathcal{M}$, is a set $\mathcal{N} \subset \mathcal{M}$ which inherits a manifold structure from $\mathcal{M}$.

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- An **embedding** map $\varphi : \mathcal{N} \to \mathcal{M}$ which is injective and structure preserving;
- The restriction $\varphi : \mathcal{N} \to \varphi(\mathcal{N})$ is a diffeomorphism.

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Rigoruous definition of submanifold:

- In terms of the above concepts, a submanifold $\mathcal{N}$ is the image $\varphi(\mathcal{N}) \subset \mathcal{M}$ of a k-dimensional manifold $(k < n)$.
- Very often it is convenient to identify $\mathcal{N}$ with $\varphi(\mathcal{N})$.
- In what follows we will mosty be concerned with 3-dimensional submanifolds. It is customary to call these **hypersurfaces**.

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Foliations

Globally hyperbolic spacetimes:

- In what follows, we assume that the spacetime $(\mathcal{M}, g_{ab})$ is **globally hyperbolic**.
- That is, we assume that its topology is that of $R \times \mathcal{S}$, where $\mathcal{S}$ is an orientable 3-dimensional manifold.
- Globaly hyperbolic spacetimes are the natural class of spacetimes on which to formulate a Cauchy problem.

Definition of a foliation:

- A spacetime is said to be **foliated** by (non-intersecting) hypersurfaces $\mathcal{S}_t$, $t \in R$ if
$$\mathcal{M} = \bigcup_{t\in\mathbb{R}} \mathcal{S}_t,$$
where we identify the leaves $\mathcal{S}_t$ with $\{t\} \times \mathcal{S}$.
- It is customary to think of the hypersurface $\mathcal{S}_0$ as an initial hypersurface on which the initial information giving rise to the spacetime is to be prescribed.

Time functions

Definition:

- In what follows it will be convenient to assume that the hypersurfaces $\mathcal{S}_t$ arise as the level surfaces of of a scalar function t which will be interpreted as a **global time function**.
- From t one can define the the covector

$$\omega_a = \nabla_a t.$$

By construction ω_a denotes the normal to the leaves $\mathcal{S}_t$ of the foliation.

- The covector ω_a is *closed* —that is,

$$\nabla_{[a}\omega_{b]} = \nabla_{[a}\nabla_{b]}t = 0.$$

The lapse function

Definition:

- Fom ω_a one defines a scalar α called the **lapse function** via

$$g^{ab}\nabla_a t \nabla_b t = \nabla^a t \nabla_a t \equiv -1/\alpha^2.$$

- The lapse measures how much proper time elapses between neighbouring time slices along the direction given by the normal vector $\omega^a \equiv g^{ab}\omega_b$.
- Assume that $\alpha > 0$ so that ω^a. Notice that ω^a is assumed to be timelike so that the hypersurfaces $\mathcal{S}_t$ are spacelike.

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Unit normal:

- In what follows we define the **unit normal** n_a via

$$n_a \equiv -\alpha\omega_a.$$

- The minus sign in the last definition is chosen so that n^a points in the direction of increasing t.
- One can readily verify that $n^a n_a = -1$.
- One thinks of n^a as the 4-velocity of a normal observer whose worldline is always orthogonal to the hypersurfaces $\mathcal{S}_t$.

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The intrinsic metric (I)

Definition:

- The spacetime metric g_{ab} induces a 3-dimensional Riemannian metric h_{ij} on $\mathcal{S}_t$.
- The relation between g_{ab} and h_{ab} is given by

$$h_{ab} \equiv g_{ab} + n_a n_b.$$

- In the previous formula we regard the 3-metric as an object living on spacetime.

Properties:

- The tensor h_{ab} is **purely spatial** —i.e. it has no component along n^a.
- Contracting with the normal:

$$n^a h_{ab} = n^a g_{ab} + n_a n^a n_b = n_b - n_b = 0,$$

- The inverse 3-metric h^{ab} is obtained by raising indices with

$$h^{ab} = g^{ab} + n^a n^b$$

The intrinsic metric (II)

Use as a projector:

- The 3-metric h_{ab} can be used to project all geometric objects along the direction given by n^a.
- Effectively, h_{ab} decomposes tensors into a **purely spatial part** which lies on the hypersurfaces $\mathcal{S}_t$ and a **timelike part** normal to the hypersurface.
- In actual computations it is convenient to consider

$$h_a{}^b = \delta_a{}^b + n_a n^b.$$

- Given a tensor T_{ab} its spatial part, to be denoted by $T^{\perp}_{ab}$ is defined to be

$$T^{\perp}_{ab} \equiv h_a{}^c h_b{}^d T_{cd}.$$

The normal projector

Definition:

- One can also define a *normal projector* ${N_a}^b$ as

$$
{N_a}^b \equiv -n_a n^b = {\delta_a}^b - {h_a}^b.
$$

- In terms of these operators an arbitrary projector can be decomposed as

$$
v^a = \delta^a{}_b v^b = ({h_a}^b + {N_a}^b) = v^{\perp a} - n^a n_b v^b.
$$

Covariant derivatives on hypersurfaces (I)

A definition of a covariant drivative:

- The 3-metric h_{ij} defines in a unique manner a covariant derivative D_i —the Levi-Civita connection of h_{ij}.
- Work from a 4-dimensional (spacetime) perspective so that we write D_a.
- One requires D_a to be torsion-free and compatible with the metric h_{ab}.
- For a scalar ϕ

  $$D_a\phi \equiv h_a{}^b\nabla_b\phi,$$

  and, say, for a $(1,1)$ tensor

  $$D_a T^b{}_c \equiv h_a{}^d h_e{}^b h_c{}^f \nabla_d T^e{}_f,$$

  with an obvious extension to other tensors.
- In coordinates, the covariant derivative D_a is associated to the **spatial Christoffel symbols**

  $$\gamma^\mu{}_{\nu\lambda} = \frac{1}{2}h^{\mu\rho}(\partial_\nu h_{\rho\lambda} + \partial_\lambda h_{\nu\rho} - \partial_\rho h_{\nu\lambda}).$$

Covariant derivatives on hypersurfaces (II)

The curvature of D_a:

- Being a covariant derivative, one can naturally associate a curvature tensor $r^a{}_{bcd}$ to D_a by considering its commutator:

$$D_a D_b v^c - D_b D_a v^c = r^c{}_{dab} v^d$$

One can verify that $r^c{}_{dab} n^d = 0$.

- Similarly, one can define the Ricci tensors and scalar as

$$r_{db} \equiv r^c{}_{dcb}, \qquad r \equiv g^{ab} r_{ab}.$$

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The extrinsic curvature (I)

Motivation:

- The Einstein field equation $R_{ab} = 0$ imposes some conditions on the 4-dimensional Riemann tensor $R^a{}_{bcd}$.
- In order to understand the implications of the Einstein equations on an hypersurface one needs to decompose $R^a{}_{bcd}$ into spatial parts. This decomposition naturally involves $r^a{}_{bcd}$.
- The tensor $r^a{}_{bcd}$ measures the **intrinsic curvature** of the hypersurface $\mathcal{S}_t$. This tensor provides no information about how $\mathcal{S}_t$ fits in $(\mathcal{M}, g_{ab})$.
- The missing information is contained in the so-called **extrinsic curvature**.

The extrinsic curvature (II)

Definition:

- The extrinsic curvature is defined as the following projection of the spacetime covariant derivative of the normal to $\mathcal{S}_t$:

$$K_{ab} \equiv -h_a{}^c h_b{}^d \nabla_{(c} n_{d)} = -h_a{}^c h_b{}^d \nabla_c n_d.$$

  The second equality follows from the fact that n_a is **rotation free**.
- By construction the extrinsic curvature is symmetric and purely spatial.
- It measures how the normal to the hypersurface changes from point to point.
- It also measures the rate at which the hypersurface deforms as it is carried along the normal **—Ricci identity**.

The acceleration

Definition:

- The *acceleration* of a foliation is define via

$$a_a \equiv n^b \nabla_b n_a.$$

- Using $n^d \nabla_c \nabla_d = 0$, one can compute

$$\begin{aligned} K_{ab} &= -h_a{}^c h_b{}^d \nabla_c n_d \\ &= -(\delta_a{}^c + n_a n^c)(\delta_b{}^d + n_b n^d) \\ &= -(\delta_a{}^c + n_a n^c)\delta_b{}^d \nabla_c n_d \\ &= -\nabla_a n_b - n_a a_b. \end{aligned}$$

An alternative expression for the extrinsic curvature

The Lie derivative of the intrinsic metric:

- One computes

$$\begin{aligned}\mathcal{L}_n h_{ab} &= \mathcal{L}_n (g_{ab} + n_a n_b) \\ &= 2\nabla_{(a} n_{b)} + n_a \mathcal{L}_n n_b + n_b \mathcal{L}_n n_a \\ &= 2(\nabla_{(a} n_{b)} + n_{(a} a_{b)}) \\ &= -2K_{ab}.\end{aligned}$$

Mean curvature

Definition:

- A related object is the so-called **mean curvature**:

$$K \equiv g^{ab}K_{ab} = h^{ab}K_{ab}.$$

- One can compute (exercise):

$$K = -\mathcal{L}_n(\ln \det h).$$

- Thus the mean curvature measures the fractional change in 3-dimensional volume along the normal n^a.
- An hypersuface for which $K = 0$ everywhere is called **maximal** —it encloses maximum volume for a given area.

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The Gauss-Codazzi equation

Motivation:

- Given the extrinsic curvature of an hypersurface $\mathcal{S}_t$, we now look how this relates to the curvature of spacetime.
- A computation using the definitions of the previous section shows that

$$D_a D_b v^c = h_a{}^p h_b{}^q h_r{}^c \nabla_p \nabla_q v^r - K_{ab} h_r{}^c n^p \nabla_p v^r - K_a{}^c K_{bp} v^p.$$

- Combining with the commutator

$$D_a D_b v^c - D_b D_a v^c = r^c{}_{dab} v^d,$$

  after some manipulations one obtains

$$r_{abcd} + K_{ac} K_{bd} - K_{ad} K_{cb} = h_a{}^p h_b{}^q h_c{}^r h_d{}^s R_{pqrs}.$$

- This equation is called the **Gauss-Codazzi equation**. It relates the spatial projection of the spacetime curvature tensor to the 3-dimensional curvature.

The Codazzi-Mainardi equation

Motivation:

- A further important identity arises from considering projections of R_{abcd} along the normal direction. This involves a spatial derivative of the extrinsic curvature.
- One has that

$$D_a K_{bc} = h_a{}^p h_b{}^q h_c{}^r \nabla_p K_{qr}.$$

- From this expression after some manipulations one can deduce

$$D_b K_{ac} - D_a K_{bc} = h_a{}^p h_b{}^q h_c{}^r n^s R_{pqrs}.$$

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In the sequel:

- In the sequel, we explore the consequences of the Gauss-Codazzi and Codazzi-Mainardi equations for the initial value problem in General Relativity.
- These give rise to the so-called **constraint equations of General Relativity**.

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The constraint equations

Strategy:

- The $3+1$ decomposition of the Einstein field equations allows to identify the intrinsic metric and the extrinsic curvature of an initial hypersurface $\mathcal{S}_0$ as the **initial data** to be prescribed for the evolution equations of General Relativity.
- In what follows we will make use of the **Gauss-Codazzi** and the **Codazzi-Mainardi** equations to extract the consequences of the vacuum Einstein field equations

$$R_{ab} = 0$$

on a hypersurface $\mathcal{S}$.

The Hamiltonian constraint (I)

Derivation of the equation:

- Contracting the Gauss-Codazzi equation one finds that

$$h^{pr}h_b{}^q h_d{}^s R_{pqrs} = r_{bd} + KK_{bd} - K^c{}_d K_{cb},$$

where $K \equiv h^{ab}K_{ab}$ denotes the trace of the extrinsic curvature.

- A further contraction then yields

$$h^{pr}h^{qs}R_{pqrs} = r + K^2 - K_{ab}K^{ab}.$$

- Now, the left-hand side can be expanded into

$$\begin{aligned} h^{pr}h^{qs}R_{pqrs} &= (g^{pr} + n^p n^s)(g^{qs} + n^q n^s) \\ &= R + 2n^p n^r R_{pr} + n^p n^r n^q n^s R_{pqrs} = 0. \end{aligned}$$

The last term vanishes beacuse of the symmetries of the Riemann tensor.

The Hamiltonian constraint (II)

Summarising

Combining the equations from the previous calculations one obtains the so-called **Hamiltonian constraint**:

$$r + K^2 - K_{ab}K^{ab} = 0.$$

The momentum constraint

Derivation:

- Contracting once the Codazzi-Mainardi equation one has that

$$D^b K_{ab} - D_a K = h_a{}^p h^{qr} n^s R_{pqrs}.$$

- The right hand side of this equation can be, in turn, expanded as

$$\begin{aligned} h_a{}^p h^{qr} n^s R_{pqrs} &= -h_a{}^p (g^{qr} + n^p n^r) n^s R_{qprs} \\ &= -h_a{}^p n^s R_{ps} - h_a{}^p n^q n^r n^s R_{pqrs} = 0, \end{aligned}$$

where in the last equatlity one makes use, again, of the vacuum Equations and the symmetries of the Riemann tensor.

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Summarising:

Combining the previous expressions one obtains the so-called **momentum constraint**:

$$D^b K_{ab} - D_a K = 0.$$

Initial data and the constraint equations

Discussion:

- The Hamiltonian and momentum constraint involve only the 3-dimensional intrinsic metric, the extrinsic curvature and their spatial derivatives.
- They are the conditions that allow a 3-dimensional slice with data (h_{ab}, K_{ab}) to be embedded in a 4-dimensional spacetime $(\mathcal{M}, g_{ab})$.
- The existence of the constraint equations implies that the data for the Einstein field equations **cannot be prescribed freely**.

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Remark:

An important point still to be clarified is the sense in which the fields h_{ab} and K_{ab} correspond to data for the Einstein field equations. To see this, one has to analyse the evolution equations implied by the Einstein field equations.

The constraint equations for the electromagnetic field (I)

A source of insight:

- The equations of other physical theories also imply constraint equations. The classical example in this respect is given by the Maxwell equations.
- In order to analyse the constraint equations implied by the Maxwell equations it is convenient to introduce the **electric** and **magnetic parts** of the Faraday tensor F_{ab}:

$$E_a \equiv F_{ab} n^b, \qquad B_a \equiv \frac{1}{2}\epsilon_{ab}{}^{cd} F_{cd} n^b = F^*_{ab} n^b.$$

- A calculation then shows that the Maxwell equations imply the constraint equations

$$D^a E_a = 0, \qquad D^a B_a = 0.$$

These constraints correspond to the well-known **Gauss laws for the electric and magnetic fields**.

The constraint equations for the electromagnetic field (II)

Summarising:

- Thus, it follows that data for the Maxwell equations **cannot be prescribed freely**. The initial value of the electric and magnetic parts of the Faraday tensor must be divergence free.
- Notice, by contrast that the wave equation for a scalar field ϕ implies no constraint equations. Thus, the data for this equation **can be prescribed freely**.

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The Ricci equation

Strategy:

- In a previous lecture we have seen that the Einstein equations imply a wave equation for the components of the metric tensor. These equations are **second order**.
- In order to obtain to obtain evolution equations which are of **first order** one needs a geometric identity relating the Lie derivative of the extrinsic curvature in the direction to the normal of the foliation.

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Derivation:

- Starting from

$$\mathcal{L}_n K_{ab} = n^c \nabla_c K_{ab} + 2K_{c(a}\nabla_{b)} n^c,$$

some manipulations (see the notes) lead the so-called **Ricci equation:**

$$\mathcal{L}_a K_{ab} = n^d n^c h_a{}^q h_b{}^r R_{drcq} - \frac{1}{\alpha} D_a D_b \alpha - K_b{}^c K_{ac}.$$

- Geometrically, this equation relates the derivative of the extrinsic curvature in the normal direction to an hypersurface $\mathcal{S}$ to a time projection of the the Riemann tensor.

The time vector and the shift vector (I)

The time vector:

- The discussion from the previous paragraphs suggests that the Einstein field equations will imply an **evolution** of the data (h_{ab}, K_{ab}).
- Assumed that the spacetime $(\mathcal{M}, g_{ab})$ is foliated by a time function t whose level surfaces corresponds to the leaves of the foliation.
- Recalling that $\omega_a = \nabla_a t$, we consider now a vector t^a (the **time vector**) such that

$$t^a = \alpha n^a + \beta^a, \qquad \beta_a n^a = 0.$$

The time vector and the shift vector (II)

The shift vector:

- The vector β^a is called the **shift vector**.
- The time vector t^a will be used to **propagate coordinates** from one time slice to another.
- In other words, t^a connects points with the same spatial coordinate —hence, the shift vector measures the amount by which the spatial coordinates are shifted within a slice with respect to the normal vector.

The time vector and the shift vector (III)

Gauge functions:

- Together, the lapse and shift determine how coordinates evolve in time. The choice of these functions is fairly arbitrary and hence they are known as **gauge functions**.
- The lapse function reflects the to choose the sequence of time slices, pushing them forward by different amounts of proper time at different spatial points on a slice —this idea is usually known as the **many-fingered nature of time**.
- The shift vector reflects the freedom to relabel spatial coordinates on each slices in an arbitrary way.
- Observers **at rest** relative to the slices follow the normal congruence n^a and are called **Eulerian observers**, while observers following the congruence t^a are called **coordinate observers**.
- It is observed that as a consequence of the previous definitions one has that $t^a \nabla_a t = 1$ so that the integral curves of t^a are naturally parametrised by t.

The evolution equation for the 3-metric

Derivation of the equation:

- Recalling that

$$K_{ab} = -\tfrac{1}{2}\mathcal{L}_n h_{ab}$$

and using the equation $t^a = \alpha n^a + \beta^a$ one concludes that

$$\mathcal{L}_t h_{ab} = -2\alpha K_{ab} + \mathcal{L}_\beta h_{ab},$$

where it has been used that

$$\mathcal{L}_t h_{ab} = \mathcal{L}_{\alpha n+\beta} h_{ab} = \alpha \mathcal{L}_n h_{ab} + \mathcal{L}_\beta h_{ab}.$$

- This equation will be interpreted as an evolution equation for the intrinsic metric h_{ab}.

Evolution equation for the second fundamental form

Derivation of the equation:

- In order to construct a similar equation for the extrinsic curvature one makes use of the Ricci equation.
- It is noticed that

$$\begin{aligned} n^d n^c h_a{}^q h_b{}^r R_{drcq} &= h^{cd} h_a{}^q h_b{}^r R_{drcq} - h_a{}^q h_b{}^r R_{rq} \\ &= h^{cd} h_a{}^q h_b{}^r R_{drcq}, \end{aligned}$$

  where to obtain the second equality $R_{ab} = 0$ has been used. The remaining term, $h^{cd} h_a{}^q h_b{}^r R_{drcq}$ is dealt with using the Gauss-Codazzi equation.
- Finally, noticing that

$$\mathcal{L}_t K_{ab} = \mathcal{L}_{\alpha n + \beta} K_{ab} = \alpha \mathcal{L}_n K_{ab} + \mathcal{L}_\beta K_{ab},$$

  one concludes that

$$\mathcal{L}_t K_{ab} = -D_a D_b \alpha + \alpha (r_{ab} - 2K_{ac} K^c{}_b + K K_{ab}) + \mathcal{L}_\beta K_{ab}.$$

  This is the desired evolution equation for K_{ab}.

The 3+1 equations and the Einstein field equations

Remarks:

- The evolution equations deduced in the previous slices determine the evolution of the data (h_{ab}, K_{ab}). These equations are usually known as the **ADM (Arnowitz-Deser-Misner) equations**.
- Together with the constraint equations they are **completely equivalent** to the vacuum Einstein field equations.
- The ADM evolution equations are first order equations —contrast with the wave equation for the components of the metric g_{ab} discussed in a previous lecture. However, the equations are not hyperbolic!
- Thus, one cannot apply directly the standard PDE theory to assert existence of solutions. Nevertheless, there are some more complicated versions which do have the hyperbolicity property.

The Maxwell evolution equations

A source of insight:

- As in the case of the constraint equations, it is useful to compare with the Maxwell field equations.
- Making use of the electric and magnetic part of the Faraday tensor, a computation of $\mathcal{L}_t E_a$ and $\mathcal{L}_t B_a$ together with the Maxwell equations allows to show that

$$\mathcal{L}_t E_a = \epsilon_{abc} D^b E^c + \mathcal{L}_\beta E_a,$$
$$\mathcal{L}_t B_a = -\epsilon_{abc} D^b B^c + \mathcal{L}_\beta B_a.$$

- Notice the similarity with the ADM equations!