Mathematical problems of General Relativity
Problem sheet 1

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1. Show that the Levi-Civita connection is characterised in a unique manner by the conditions:
   (a) $\nabla_a \nabla_b \phi = \nabla_b \nabla_a \phi$ (torsion-freeness);
   (b) $\nabla_a g_{bc} = 0$ (metric compatibility).

2. Show that from the rule for the covariant derivative of a vector v^a in local coordinates
$$\nabla_\mu v^\nu = \partial_\mu v^\nu + \Gamma^\nu{}_{\lambda\mu} v^\lambda,$$
and the assumptions that ∇_a satisfies the Leibinitz rule and that $\nabla_\mu \phi = \partial_\mu \phi$ with ϕ as scalar field, it follows that
$$\nabla_\mu \omega_\nu = \partial_\mu \omega_\nu - \Gamma^\lambda{}_{\nu\mu} \omega_\lambda,$$
where ω_μ are the components of a covector ω_a in local coordinates.

3. Let $R^a{}_{bcd}$ denote the Riemann tensor of the Levi-Civita connection ∇_a and let v^a be an arbitrary vector. Show that from
$$\nabla_a \nabla_b v^c - \nabla_b \nabla_a v^c = R^c{}_{dab} v^d$$
it follows that for a covector ω_a one has that
$$\nabla_a \nabla_b \omega_c - \nabla_b \nabla_a \omega_c = -R^d{}_{cab} \omega_d.$$
Hint: compute the commutator of $\omega_c v^c$.

4. Given the Riemann tensor of the Levi-Civita connection ∇_a, show that the symmetry
$$R_{abcd} = R_{cdab}$$
follows from the symmetries
$$R_{abcd} = -R_{bacd} = -R_{abdc}, \qquad R_{abcd} + R_{acdb} + R_{adbc} = 0.$$

5. Show that for the Levi-Civita connection of a metric g_{ab} the second Bianchi identity implies that
$$\nabla^a (R_{ab} - \tfrac{1}{2} R g_{ab}) = 0.$$
That is, show that the Einstein tensor is divergence-free.

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6. Show that the expression

$$\mathcal{L}_v T^\mu{}_{\lambda\rho} = v^\sigma \partial_\sigma T^\mu{}_{\lambda\rho} - \partial_\sigma v^\mu T^\sigma{}_{\lambda\rho} + \partial_\lambda v^\sigma T^\mu{}_{\sigma\rho} + \partial_\rho v^\sigma T^\mu{}_{\lambda\sigma},$$

for the Lie derivative of a $(1,2)$-tensor can be rewritten as

$$\mathcal{L}_v T^\mu{}_{\lambda\rho} = v^\sigma \nabla_\sigma T^\mu{}_{\lambda\rho} - \nabla_\sigma v^\mu T^\sigma{}_{\lambda\rho} + \nabla_\lambda v^\sigma T^\mu{}_{\sigma\rho} + \nabla_\rho v^\sigma T^\mu{}_{\lambda\sigma},$$

where ∇_μ denotes the expression of the covariant derivative ∇_a in local coordinates. Use this expression to conclude that $\mathcal{L}_v T^\mu{}_{\lambda\rho}$ corresponds to the expression in local coordinates of a tensor $\mathcal{L}_v T^a{}_{bc}$.

7. Let ξ^a denote a vector field over a manifold $\mathcal{M}$. Show that $\mathcal{L}_\xi g_{ab} = 0$ implies the Killing equation

$$\nabla_a \xi_b + \nabla_b \xi_a = 0.$$

8. Show that the vacuum Einstein field equation

$$R_{ab} - \tfrac{1}{2} R g_{ab} + \lambda g_{ab} = 0$$

can be rewritten as

$$R_{ab} = \lambda g_{ab}.$$

9. Show that if $R_{ab} = 0$ then a Killing vector satisfies the wave equation

$$\Box \xi^a = 0.$$

10. Use the identity $\partial_c \sqrt{-\det g} = \frac{1}{2}\sqrt{-\det g}\, g^{ab}\partial_c g_{ab}$ to show that the wave equation $\Box\phi = 0$ can be rewritten as

$$\Box\phi = \frac{1}{\sqrt{-\det g}} \partial_\mu \left(\sqrt{-\det g}\, g^{\mu\nu} \partial_\nu \phi \right).$$