# Mathematical problems of General Relativity
(LTCC course)

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## 1 Introduction

General Relativity is described as the flagship of Mathematical Physics. The study of the mathematical properties of the solutions to the equations of General Relativity —the Einstein field equations— has experienced a great development in recent years. Work in this area has been based on a systematic use of the so-called initial value problem for the Einstein field equations. As such, it requires the use of ideas and techniques from various branches of Mathematics —especially Differential Geometry and Partial Differential Equations (elliptic and hyperbolic). Current mathematical challenges in the area include the analysis of the global existence of solutions to the Einstein field equations, the uniqueness of stationary black holes, the non-linear stability of the Kerr spacetime, and the construction of initial data sets of geometrical or physical interest.

The main objective of these notes is to present a discussion of General Relativity as an initial value problem.

## 2 An review of Differential Geometry

The natural language of General Relativity is that of *Differential Geometry.* These notes start with a general overview of its key ideas.

### 2.1 Manifolds

The basic concept in Differential Geometry is that of a *differentiable manifold* (or manifold for short). A rigorous definition will not be presented here —the interested reader is referred to e.g. [4]. In broad terms, a manifold $\mathcal{M}$ is a topological space that can be covered by a collection of *charts* $(\mathcal{U},\phi)$ where $\mathcal{U}\subset\mathcal{M}$ is an open subset and $\phi:\mathcal{U}\rightarrow\mathbb{R}^n$ for some $n$ is a smooth injective mapping. In what follows, for simplicity and unless otherwise stated, it is assumed that all structures are smooth. The notion of a manifold requires certain compatibility between overlapping charts. Given $p\in\mathcal{U}$ one writes

$$\phi(p)=(x^1,\ldots,x^n)$$

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and the $(x^\mu) = (x^1, \ldots, x^n)$ are called the *local coordinates* on $\mathcal{U}$. An important property for a manifold to possess is *orientability*. A manifold $\mathcal{M}$ is said to be orientable if the Jacobian of the transformation between overlapping charts is positive. In these notes, attention will be restricted to manifolds of dimensions 4 and 3.

A scalar field over $\mathcal{M}$ is a smooth function $f : \mathcal{M} \to \mathbb{R}$. The set of scalar fields over $\mathcal{M}$ will be denoted by $\mathfrak{X}(\mathcal{M})$.

## 2.2 Curves and vector fields

A curve is a smooth map $\gamma : I \to \mathcal{M}$ with $I \subset \mathbb{R}$ an interval. In terms of coordinates $(x^\mu)$ defined over a chart of $\mathcal{M}$ one writes the curve as

$$x^\mu(\lambda) = (x^1(\lambda), \ldots, x^n(\lambda)),$$

where $\lambda \in I$ is the parameter of the curve.

Closely related to the notion of a curve is the concept of a *tangent vector*. It formalises the physical notion of velocity. In local coordinates, the tangent vector to the curve $x^\mu(\lambda)$ is given by

$$v^\mu = \frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}.$$

In modern Differential Geometry one identifies vectors with homogeneous first order differential operators acting of scalar fields over $\mathcal{M}$. This approach allows to encode in a simple manner the "classical" transformation properties of vectors. Following this perspective, in local coordinates a vector field will be written as $v^\mu \partial_\mu$.

In what follows, we will make use of the *abstract index notation* to denote vectors and tensors. A generic vector will in this formalism denoted as $v^a$. The role of the superindex in this notation is to indicate the character of the object in question. When we refer to the components of $v^a$ in some coordinate system $(x^\mu)$ we will write $v^\mu$ —i.e. greek indices will be used.

The set of vectors at a point $p$ of $\mathcal{M}$ is called the *tangent space* $T_p\mathcal{M}$. A smooth prescription of a vector at every point of $\mathcal{M}$ is called a *vector field*. The collection of all tangent spaces on $\mathcal{M}$ is called the *tangent bundle* and will be denoted by $T\mathcal{M}$.

## 2.3 Covectors and higher rank tensors

When woking with vectors it is natural to consider functions of vectors. A real-valued linear function of a vector is called a *covector* (or 1-form). Using abstract index notation a covector will be denoted by $\omega_a$. The action of $\omega_a$ on $v^a$ will be denoted by $\omega_a v^a$. Notice that if $\omega_a$ and $v^a$ are, respectively, covector and vector fields over $\mathcal{M}$, then $\omega_a v^a \in \mathfrak{X}(\mathcal{M})$. The set of covectors at a point $p \in \mathcal{M}$ is called the *cotangent space* $T^*\mathcal{M}$. The set of all cotangent spaces on $\mathcal{M}$ is the *cotangent bundle* $T^*\mathcal{M}$.

Given the notions of vectors and covectors, higher rank objects (tensors) can be constructed by analogy. A vector can be thought as a real-valued function of a covector —that is, the definitions of vectors and covectors are dual. This idea can be generalised so as to consider real-valued functions of $m$ covectors and $n$ vectors that are linear in all their arguments. This object will be known as a tensor of type $(m, n)$. For example, using abstract index notation, a tensor $T^{ab}{}_c$ is of type $(2, 1)$. Traditionally, "upper" indices in a tensor are called *contravariant* while "down" ones are called *covariant*.

A tensor is *symmetric* if it remains unchanged under the interchange of two of its arguments —i.e. $T_{ab} = T_{ba}$. Similarly, a tensor is *antisymmetric* if it changes sign with an interchange of a pair of arguments as in $S_{abc} = -S_{acb}$. The symmetric and antisymmetric parts of a tensor can be constructed by adding together all possible permutations with the appropriate signs: positive for the symmetric part and positive or negative for the antisymmetric part depending on whether the permutation is even or odd. For example

$$T_{(ab)} \equiv \frac{1}{2}(T_{ab} + T_{ba}), \qquad T_{[ab]} \equiv \frac{1}{2}(T_{ab} - T_{ba}).$$

## 2.4 Manifolds with metrics

A *metric* on $\mathcal{M}$ is a symmetric $(0,2)$ tensor field $g_{ab}$. The metric is said to be *non-degenerate* if $g_{ab}u^a v^b = 0$ for all $u^a$ if and only if $v^a = 0$. In what follows, we will only consider non-degenerate metrics. The metric allows to encode the geometric notions of orthogonality and norm of a vector. The norm of a vector is given by $|v|^2 \equiv g_{ab}v^a v^b$. Moreover, if $g_{ab}v^a u^a = 0$, then $v^a$ and $u^a$ are said to be *orthogonal*. In terms of a coordinate system $(x^\mu)$ the components of $g_{ab}$ —to be denoted by $g_{\mu\nu}$ can be regarded as a $n \times n$ matrix. Because this matrix is symmetric one has $n$ real eigenvalues. The signature of $g_{ab}$ is the difference between the number of positive and negative eigenvalues. If the signature is $\pm n$ then one has a *Riemannian metric*. If the signature is $\pm(n-2)$ then the metric is said to be *Lorentzian*.

A metric $g_{ab}$ can be used to define a one-to-one correspondence between vectors and covectors. In local coordinates denote by $g^{\mu\nu}$ the inverse of $g_{\mu\nu}$. This defines a $(2,0)$ tensor which we denote by $g^{ab}$. By construction $g_{ab}g^{bc} = \delta_a{}^c$ where $\delta_a{}^c$ is the *Kroneker delta*. Given a vector $v^a$ one defines $v_a \equiv g_{ab}v^a$. Similarly, given a covector $\omega_a$ one can define $\omega^a \equiv g^{ab}\omega_b$.

### 2.4.1 Remarks involving Lorentzian metric

In these notes all Lorentzian metrics will live on a 4-dimensional manifold and will be assumed to have signature 2 —that is, one has on negative eigenvalue and 3 positive ones. A Lorentzian metric can be used to classify vectors according to the sign of their norm. A vector $v^a$ is said to be *timelike*, *null* or *spacelike* according to whether $g_{ab}v^a v^b$ is negative, zero or positive.

## 2.5 Covariant derivatives

A *covariant derivative* is a notion of differentiation with tensorial properties. A precise definition will not be discussed here. A metric $g_{ab}$ allows to define a canonical covariant derivative $\nabla_a$ over $\mathcal{M}$ —the so-called *Levi-Civita connection*. The covariant derivative of a vector $v^a$ is denoted by $\nabla_a v^b$. Similarly, for a covector $\omega_b$ one writes $\nabla_a \omega_b$. Explicit formulae in terms of local coordinates involve the so-called *Christoffel symbols*

$$\Gamma^\mu{}_{\nu\lambda} = \frac{1}{2} g^{\mu\rho}(\partial_\nu g_{\rho\lambda} + \partial_\lambda g_{\nu\rho} - \partial_\rho g_{\nu\lambda}).$$

Notice that $\Gamma^\mu{}_{\nu\lambda} = \Gamma^\mu{}_{\lambda\nu}$. The Christoffel symbols do not define a tensor. In a neighbourhood of any $p \in \mathcal{M}$ there is a coordinate system (*normal coordinates*) in which the components of the Christoffel symbols vanish at $p$. In terms of the Christoffel one defines the components of $\nabla_a v^b$ as

$$\nabla_\mu v^\nu \equiv \partial_\mu v^\nu + \Gamma^\nu{}_{\lambda\mu} v^\lambda.$$

From this expression the covariant derivative of a covector $\omega_a$ can be deduced:

$$\nabla_\mu \omega_\nu = \partial_\mu \omega_\nu - \Gamma^\lambda{}_{\nu\mu}\omega_\lambda.$$

More generally, one has, for example, that

$$\nabla_\mu T^\nu{}_{\lambda\rho} = \partial_\mu T^\nu{}_{\lambda\rho} + \Gamma^\nu{}_{\sigma\mu} T^\sigma{}_{\lambda\rho} - \Gamma^\sigma{}_{\lambda\mu} T^\nu{}_{\sigma\rho} - \Gamma^\sigma{}_{\rho\mu} T^\nu{}_{\lambda\sigma}.$$

The Levi-Civita connection is defined in such a way that $\nabla_a g_{bc} = 0$ —as it can be verified by an explicit computation.

An important class of curves is given by *geodesics*. If $v^a$ denotes the tangent vector to a curve $\gamma : I \to \mathcal{M}$, then the curve is a (metric) geodesic if and only if

$$v^a \nabla_a v^b = f v^b,$$

with $f$ some function of the parameter of the curve $\lambda$. In the case $f = 0$, the parameter is called *affine*. More generally, a vector field $u^a$ defined along a curve $\gamma$ with tangent $v^a$ is said to be *parallely transported* along $\gamma$ if $v^a \nabla_a u^b = 0$. Thus, an affinely parametrised geodesic is precisely one whose tangent vector is parallely transported along itself. An affine parameter is unique up to an affine transformation $\lambda \mapsto a\lambda + b$ for constants $a$ and $b$.

## 2.6 Curvature

In this section it is assumed that $\nabla_a$ is the *Levi-Civita* connection of a metric $g_{ab}$. The notion of curvature arises in a natural way from the *commutator* of covariant derivatives acting on a vector $v^a$. More precisely, one has that

$$\nabla_a \nabla_b v^c - \nabla_b \nabla_a v^c = R^c{}_{dab} v^d, \tag{1}$$

where $R^c{}_{dab}$ is the *Riemann curvature tensor*. From equation (1) one can deduce the corresponding commutator of covariant derivatives for a covector. Namely, one finds that

$$\nabla_a \nabla_b \omega_c - \nabla_b \nabla_a \omega_c = -R^d{}_{cab} \omega_d.$$

These expressions can be generalised in a natural way to higher rank tensors.

In terms of local coordinates $(x^\mu)$ the components of the Riemann tensors can be written as

$$R^\mu{}_{\nu\lambda\rho} = \partial_\lambda \Gamma^\mu{}_{\nu\rho} - \partial_\rho \Gamma^\mu{}_{\nu\lambda} + \Gamma^\mu{}_{\lambda\sigma} \Gamma^\sigma{}_{\nu\rho} - \Gamma^\mu{}_{\rho\sigma} \Gamma^\sigma{}_{\nu\lambda}.$$

Taking traces of $R^a{}_{bcd}$ one defines the *Ricci tensor* $R_{bd} \equiv R^a{}_{bad}$ and *Ricci scalar* $R \equiv g^{ab} R_{ab}$. It is also customary to define the *Einstein tensor*

$$G_{ab} = R_{ab} - \frac{1}{2} R g_{ab}.$$

The Riemann tensor satisfies the following symmetries:

$$\begin{aligned} &R_{abcd} = -R_{bacd} = -R_{abdc} = R_{badc}, \\ &R_{abcd} = R_{cdab}, \\ &R_{abcd} + R_{acdb} + R_{adbc} = 0. \end{aligned}$$

The last of these identities is known as the *first Bianchi identity*. In addition, the Riemann tensor satisfies a differential identity, the *second Bianchi identity*:

$$\nabla_a R_{bcde} + \nabla_b R_{cade} + \nabla_c R_{abde} = 0.$$

Contracting twice this identity with the metric shows that $\nabla^a G_{ab} = 0$.

## 2.7 Lie derivatives

Another type of derivative on a manifold which is defined in an invariant way is the so-called *Lie derivative*. This derivative measures the change of a tensor as it is transported along the direction prescribed by a vector field $v^a$ and it is denoted by $\mathcal{L}_v$. The Lie derivative of a tensor $T^a{}_{bc}$ is given in coordinates by

$$\mathcal{L}_v T^\mu{}_{\lambda\rho} = v^\sigma \partial_\sigma T^\mu{}_{\lambda\rho} - \partial_\sigma v^\mu T^\sigma{}_{\lambda\rho} + \partial_\lambda v^\sigma T^\mu{}_{\sigma\rho} + \partial_\rho v^\sigma T^\mu{}_{\lambda\sigma},$$

and can be verified to be a tensor. Lie derivatives of other tensors can be defined in an analogous way.

# 3 Survey of General Relativity

General Relativity is a relativistic theory of gravity. It describes the gravitational interaction as a manifestation of the curvature of spacetime. One of the key tenets of General Relativity is that both matter and energy produce curvature of the spacetime. The way matter and energy produce curvature in spacetime is described by means of the *Einstein field equations*. One of the main properties of the gravitational field as described by General Relativity is that it can be a source of itself —this is a manifestation of the non-linearity of the Einstein field equations. This property gives rise to a variety of phenomena that can be analysed by means of the so-called *vacuum Einstein field equations* without having to resort to any further considerations about matter sources.

As it is the case of many other physical theories, General Relativity admits a formulation in terms of an *initial value problem* (*Cauchy problem*) whereby one prescribes the geometry of spacetime at some instant of time and then one purports to reconstruct it from the initial data. Part of the the task in the construction of the initial problem in General Relativity is to make sense of what it means to prescribe the geometry of spacetime at at instant of time. A second part of the task is to show how the spacetime is to be reconstructed from the data. The initial value problem is the core of the area of research broadly known as *mathematical Relativity* —an area of active research with a number of interesting and challenging open problems.

Before turning the attention to the initial value problem in General Relativity, it is convenient to provide a survey of the key ideas in General Relativity to see why a discussion of the Cauchy problem is necessary/helpful.

## 3.1 The Einstein field equations

General Relativity postulates the existence of a 4-dimensional manifold $\mathcal{M}$, the *spacetime manifold* which contains events as points. This spacetime manifold is endowed with a Lorentzian metric $g_{ab}$ which in these lectures is assumed to have signature +2 —i.e. $(-+++)$. By a spacetime it will understood the a pair $(\mathcal{M}, g_{ab})$ where the metric $g_{ab}$ satisfies the Einstein field equations

$$R_{ab} - \frac{1}{2}Rg_{ab} + \lambda g_{ab} = T_{ab}. \tag{2}$$

In the previous equation $\lambda$ denotes the so-called *Cosmological constant* while $T_{ab}$ is the *energy-momentum tensor* of the matter model under consideration and it encodes the information about the matter. The main goal of *mathematical General Relativity* is to obtain a qualitative understanding of the solutions to the Einstein field equations.

The conservation of energy-momentum is encoded in the condition

$$\nabla^a T_{ab} = 0. \tag{3}$$

The conservation equation (3) is consistent with the Einstein field equations as a consequence of the second Bianchi identity. More precisely, one has that

$$\nabla^a \left( R_{ab} - \frac{1}{2}Rg_{ab} + \lambda g_{ab} \right) = 0.$$

In these notes the focus will be on systems describing *isolated bodies* so that henceforth we assume that $\lambda = 0$. Moreover, attention is restricted to the *vacuum* case for which $T_{ab} = 0$. The vacuum equations apply in the exterior region to an astrophysical source, but they usefulness is not restricted to this. There exist "stand alone" vacuum configurations —like for example, black holes. A direct calculation in the vacuum case with vanishing Cosmological constant allows to rewrite the Einstein field equations (2) as

$$R_{ab} = 0. \tag{4}$$

The field equations prescribe the geometry of spacetime locally. However, they do not prescribe the topology of the spacetime manifold.

The geometry of the spacetime can be probed by means of the movement of test particles: General Relativity postulates that massive test particles move along timelike geodesics while rays of light follow null geodesics.

## 3.2 Exact solutions to the Einstein field equations

Given the vacuum field equations (4), a natural question to ask is whether there are any solutions. What should one understand for a solution to equation (4)? In first instance, a solution is given by a metric $g_{ab}$ expressed in a specific coordinate system $(x^\mu)$ —in what follows we will write this as $g_{\mu\nu}$. Exact solutions are our main way of acquiring intuition about the behaviour of generic solutions to the Einstein field equations.

### 3.2.1 The Minkowski spacetime

The simplest example of a solution is given by the metric encoded in the line element

$$g = \eta_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu, \qquad \eta_{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1). \tag{5}$$

One clearly verifies that for this metric $R_{\mu\nu\lambda\rho} = 0$ so that $R_{\mu\nu} = 0$. As $R_{\mu\nu}$ are the components of a tensor in a specific coordinate system one concludes then $R_{ab} = 0$. Thus, any metric related to (5) by a coordinate transformation is a solution to the vacuum field equation (4). Accordingly, one has obtained a tensor field $g_{ab}$ satisfying the equation (4).

The previous example shows that as a consequence of the tensorial character of the Einstein field equations a solution to the equations is, in fact, an equivalence class of solutions related to each other by means of coordinate transformations.

### 3.2.2 Symmetry assumptions

In order to find further explicit solutions to equation (4) one needs to make some sort of assumptions about the spacetime. A standard assumption is that the spacetime has *continuous symmetries*. The notion of a continuous symmetry is formalised by the concept of a *diffeomorphism*. A diffeomorphism is a smooth map $\phi$ of $\mathcal{M}$ onto itself. One can think of the diffeomorphism in terms of displacements of points in the manifold along curves in the manifold —these curves are called the *orbits of the symmetry*. In what follows, let $\xi^a$ denote the tangent vector to the orbits. The mapping $\phi$ is called an isometry if $\mathcal{L}_\xi g_{ab} = 0$ —that is, if the symmetry leaves the metric invariant. One can verify that this condition implies the equation

$$\nabla_a \xi_b + \nabla_b \xi_a = 0. \tag{6}$$

This equation is called the *Killing equation*. An important observation about this equation is that it is *overdetermined* —this means that it does not admit a solution for a general spacetime. In other words, the existence of a solution imposes restrictions on the manifold. This can be best understood by considering integrability conditions for equation (6). Given the commutator

$$\nabla_a \nabla_b \xi_c - \nabla_b \nabla_a \xi_c = -R^d{}_{cab}\xi_d,$$

using equation (6) one obtains that

$$\nabla_a \nabla_b \xi_c + \nabla_b \nabla_c \xi_a = -R^d{}_{cab}\xi_d.$$

Shuffling the indices in a cyclic way one obtains the further equations

$$\nabla_c \nabla_a \xi_b + \nabla_a \nabla_b \xi_c = -R^d{}_{bca}\xi_d, \qquad \nabla_b \nabla_c \xi_a + \nabla_c \nabla_a \xi_b = -R^d{}_{abc}\xi_d.$$

Adding the first two equations and subtracting the third one one gets

$$2\nabla_a \nabla_b \xi_c = (R^d{}_{abc} - R^d{}_{cab} - R^d{}_{bca})\xi_d.$$

Finally, using the first Bianchi identity one has that $-R^d{}_{cab} = R^d{}_{abc} + R^d{}_{bca}$ so that

$$\nabla_a \nabla_b \xi_c = R^d{}_{abc}\xi_d. \tag{7}$$

This is an integrability condition for a solution to the Killing equation —i.e. a necessary condition that needs to be satisfied by a solution to (6). It shows that if one has a solution to the Killing equation, then the curvature of the spacetime is restricted.

An important type of symmetry is the so-called *spherical symmetry*. In broad terms, this means that there exists a 3-dimensional group of symmetries with 2-dimensional spacelike orbits. Each orbit is an *homogeneous* and *isotropic* manifold. The orbits are required to be compact and to have constant positive curvature.

### 3.2.3 The Schwarzschild solution

Arguably, the most important solution to the vacuum Einstein field equations is the *Schwarzschild spacetime*, given in standard coordinates $(t, r, \theta, \varphi)$ by the line element

$$g = -\left(1 - \frac{2m}{r}\right)\mathrm{d}t^2 + \left(1 - \frac{2m}{r}\right)^{-1}\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\varphi^2). \tag{8}$$

This solution is spherically symmetric and static —i.e. time independent, as it can be seen by direct inspection of the metric. In a later section we will further elaborate on the notion of static solutions. For a discussion of the interpretation and basic properties of this solution, the reader is referred to Wald's book [**?**]. Here we make some remarks which will motivate subsequent topics of this notes. The first one is that staticity can be obtained as a consequence of the assumption of spherical symmetry —this is usually called the *Birkhoff theorem*: any spherically symmetric solution to the vacuum field equations is locally isometric to the Schwarzschild solution (8). The second observation is that the Schwarzschild solution can be characterised as the only solution of the vacuum equations (4) satisfying a certain (reasonable) behaviour at infinity —*asymptotic flatness*: the requirement that asymptotically, the metric behaves like the Minkowski metric. This result is known as the *no-hair theorem*. The Birkhoff and no-hair theorems constitute examples of a type of results for solutions to the Einstein field equations known as *rigidity theorems* — these show that under assuming certain properties about solutions to the Einstein field equations immediately imply other properties.

The Schwarzschild solution is of particular interest as it gives the simplest example of a *black hole*. The spacetime manifold can be explicitly verified to be singular at $r = 0$. This singularity is hidden behind a *horizon*.

### 3.2.4 Other solutions to the Einstein field equations

A natural question is: are there other further exact solutions? The simple direct answer is in the affirmative. To obtain more solutions the natural strategy is to reduce the number of symmetries —accordingly the task of finding solutions becomes harder. A natural assumption is to look for axially symmetric and stationary solutions —stationarity is a form of time independence which is compatible with the notion of rotation. This assumption leads to the *Kerr spacetime* describing a time independent rotating black hole. The notion of stationarity will be elaborated in a later section.

At this point, the construction of solutions by means of symmetries reaches an impasse. Although there are a huge number of explicit solutions to the Einstein field equation —see e.g. the monograph [11]— the number of solutions with a physical/geometric relevance is much more restricted —for a discussion of some of the physically/geometrically important solutions see e.g. [6].

### 3.2.5 Abstract analysis of the Einstein field equations

An alternative to the analysis of solutions to Einstein field equations by means of the construction of exact solutions is to us the general features and structure of the equations to assert existence in an "abstract sense". This approach can be further employed to establish uniqueness and other properties of the solutions. This approach to Relativity has been strongly advocated in e.g. [8]. After this type of analysis has been carried out one can proceed to construct solutions numerically.

# 4 A first look at the Cauchy problem in General Relativity

A strategy to pursue the programme described in the previous paragraph is to formulate an *initial value problem* (*Cauchy problem*) for the Einstein field equations. To see what sort of issues are involved in this, it is convenient to look at a similar discussion in simpler equations.

All throughout one assumes one has a spacetime $(\mathcal{M}, g_{ab})$.

## 4.1 The initial value problem for the scalar wave equation

In first instance consider the wave equation $\Box\phi \equiv \nabla_a\nabla^a\phi = 0$ with respect to the metric $g_{ab}$. In local coordinates it can be shown that

$$\Box\phi = \frac{1}{\sqrt{-\det g}}\partial_\mu\left(\sqrt{-\det g}\, g^{\mu\nu}\partial_\nu\phi\right). \tag{9}$$

The *principal part* of the equation $\Box\phi = 0$ corresponds to the terms in (9) containing the highest derivatives of the scalar field $\phi$ —namely

$$g^{\mu\nu}\partial_\mu\partial_\nu\phi.$$

The structure in this expression is particular of a class of partial differential equations known as *hyperbolic equations*. The prototypical hyperbolic equation is the wave equation on the Minkowski spacetime. In standard Cartesian coordinates one has that

$$\Box\phi = \eta^{\mu\nu}\partial_\mu\partial_\nu\phi = \partial_x^2\phi + \partial_y^2 + \partial_z^2\phi - \partial_t^2\phi = 0.$$

The Cauchy problem for the wave equations and more general hyperbolic equations is well understood in a *local setting*. Roughly speaking this means that if one prescribes the field $\phi$ and its derivative $\partial_\mu\phi$ at some fiduciary instant of time $t = 0$, then the equation $\Box\phi = 0$ has a solution for suitably small times (*local existence*). Moreover, this solution is *unique* in its existence interval and it has *continuous dependence* on the initial data. The question of *global existence* is much more challenging and, in fact, an open issue for general spacetimes $(\mathcal{M}, g_{ab})$.

## 4.2 The Maxwell equations as wave equations

A useful model to discuss certain issues arising in the Einstein field equations are the *source-free Maxwell equations*:

$$\nabla^a F_{ab} = 0, \qquad \nabla_{[a}F_{bc]} = 0, \tag{10}$$

where $F_{ab} = -F_{ab}$ is the *Faraday tensor*. A solution to the second Maxwell equation is given by

$$F_{ab} = \nabla_a A_b - \nabla_b A_a, \tag{11}$$

where $A_a$ is the so-called *gauge potential*. This statement can be verified by means of a direct computation. The gauge potential does not determine the the Faraday tensor in a unique way as $A_a + \nabla_a\phi$ with $\phi$ as scalar field gives the same $F_{ab}$. Substituting equation (11) into the first Maxwell equation one has that

$$\begin{aligned} 0 &= \nabla^a\left(\nabla_a A_b - \nabla_b A_a\right) \\ &= \nabla^a\nabla_a A_b - \nabla^a\nabla_b A_a. \end{aligned}$$

Now, using the commutator $\nabla_a\nabla_b A_c - \nabla_b\nabla_a A_c = -R^d{}_{cab}A_d$ it follows that $\nabla^a\nabla_b A_a = \nabla_b\nabla^a A_a + R^d{}_b A_d$ so that one concludes that $A_b$ satisfies the equation

$$\nabla^a\nabla_a A_b - \nabla_b\nabla^a A_a - R^a{}_b A_a = 0. \tag{12}$$

The question is now: under what circumstances one can assert the existence of solutions to equation (12) on a smooth spacetime $(\mathcal{M}, g_{ab})$. The principal part of equation of equation (12) is given by

$$\partial^\mu\partial_\mu A_\nu - \partial_\nu\partial^\mu A_\mu.$$

The key observation is that if one could remove the second term in the principal part, one would have the same principal part as for a wave equation for the components of $A_a$. The gauge freedom of the Maxwell equations can be exploited to this end. Making the replacement $A_\nu \to A_\nu + \nabla_\nu\phi$, with $\phi$ chosen such that

$$\nabla^\mu\nabla_\mu\phi = -\nabla^\mu A_\mu \tag{13}$$

one obtains that

$$\nabla^\mu A_\mu \to \nabla^\mu A_\mu + \nabla^\mu \nabla_\mu \phi = 0.$$

Equation (13) is to be interpreted as a wave equation for $\phi$ with source term given by $-\nabla^\mu A_\mu$. One says that the gauge potential is in the *Lorenz gauge* and it satisfies the wave equation

$$\nabla^\mu \nabla_\mu A_\nu = R^\mu{}_\nu A_\mu. \tag{14}$$

In order to assert existence to the Maxwell equations one then considers the system of wave equations (13)-(14). These equations are manifestly hyperbolic so that local existence is obtained provided that suitable initial data is provided. This initial data consists of the values of $\phi$, $\nabla_\mu \phi$, $A_\nu$ and $\nabla_\mu A_\nu$ at some initial time.

### 4.3 The Einstein field equations in wave coordinates

In order to provide a first discussion of the Cauchy problem for the Einstein field equations, we start by observing that given general coordinates $(x^\mu)$, the Ricci tensor $R_{ab}$ can be explicitly written in terms of the components of the metric tensor $g_{\mu\nu}$ and its first and second partial derivatives as

$$\begin{aligned} R_{\mu\nu} = & \frac{1}{2} \sum_{\lambda,\rho=0}^{3} \left(\partial_\lambda \left(g^{\lambda\rho} \left(\partial_\mu g_{\rho\nu} + \partial_\nu g_{\mu\rho} - \partial_\rho g_{\mu\nu}\right)\right) - \partial_\nu \left(g^{\lambda\rho} \partial_\mu g_{\lambda\rho}\right)\right) \\ & + \frac{1}{4} \sum_{\lambda,\rho,\sigma,\tau=0}^{3} \Big( g^{\sigma\tau} g^{\lambda\rho} \left(\partial_\sigma g_{\rho\tau} + \partial_\rho g_{\sigma\tau} - \partial_\tau g_{\sigma\rho}\right) \left(\partial_\nu g_{\mu\lambda} + \partial_\mu g_{\lambda\nu} - \partial_\lambda g_{\mu\nu}\right) \\ & \qquad - g^{\rho\sigma} g^{\lambda\tau} \left(\partial_\nu g_{\lambda\sigma} + \partial_\lambda g_{\nu\sigma} - \partial_\sigma g_{\nu\lambda}\right) \left(\partial_\sigma g_{\mu\tau} + \partial_\mu g_{\sigma\tau} - \partial_\tau g_{\sigma\mu}\right) \Big), \end{aligned} \tag{15}$$

where $g^{\lambda\rho} = (\det g)^{-1} p^{\lambda\rho}$ with $p^{\lambda\rho}$ polynomials of degree 3 in $g_{\mu\nu}$. The summation symbols have been included in the above expression for the sake of clarity. Thus, the vacuum Einstein field equation implies a *second order quasilinear partial differential equations* for the components of the metric tensor. As (15) is a second order differential equation for $g_{\mu\nu}$ one may hope it is possible to recast it in the form of some type of wave equation. As it will be seen, this involves a coordinate specification.

By recalling the formula for the Christoffels symbols in terms of partial derivatives of the metric tensor

$$\Gamma^\nu{}_{\mu\lambda} = \frac{1}{2} g^{\nu\rho} (\partial_\mu g_{\rho\lambda} + \partial_\lambda g_{\mu\rho} - \partial_\rho g_{\mu\lambda}),$$

and by defining

$$\Gamma^\nu \equiv g^{\mu\lambda} \Gamma^\nu{}_{\mu\lambda},$$

one can rewrite the formula (15) more concisely as

$$R_{\mu\nu} = -\frac{1}{2} g^{\lambda\rho} \partial_\lambda \partial_\rho g_{\mu\nu} + \nabla_{(\mu} \Gamma_{\nu)} + g_{\lambda\rho} g^{\sigma\tau} \Gamma^\lambda{}_{\sigma\mu} \Gamma^\rho{}_{\tau\nu} + 2\Gamma^\sigma{}_{\lambda\rho} g^{\lambda\tau} g_{\sigma(\mu} \Gamma^\rho{}_{\nu)\tau}. \tag{16}$$

In this form, the principal part of the vacuum Einstein field equation (4) can be readily be identified to be given by the terms

$$-\frac{1}{2} g^{\lambda\rho} \partial_\lambda \partial_\rho g_{\mu\nu} + \nabla_{(\mu} \Gamma_{\nu)}.$$

An approach to the construction of systems of coordinates $(x^\mu)$ which, in turn, leads to a suitable hyperbolic equation for the components of the metric tensor $g_{ab}$ is to require the coordinates to satisfy the equation

$$\nabla^\nu \nabla_\nu x^\mu = 0, \tag{17}$$

where the coordinates $x^\mu$ are treated as a scalar field over $\mathcal{M}$. A direct computation then shows that

$$\nabla_\nu x^\mu = \partial_\nu x^\mu = \delta_\nu{}^\mu,$$
$$\nabla_\lambda \nabla_\nu x^\mu = \partial_\lambda \delta_\nu{}^\mu - \Gamma^\rho{}_{\lambda\nu}\delta_\rho{}^\mu = -\Gamma^\mu{}_{\nu\lambda},$$

so that

$$\nabla^\nu \nabla_\nu x^\mu = g^{\nu\lambda}\Gamma^\mu{}_{\nu\lambda} = -\Gamma^\mu. \tag{18}$$

If suitable initial data is provided for equation (18) —the coordinate differentials $\mathrm{d}x^a$ have to be chosen initially to be point-wise independent— then general theory of hyperbolic differential equations ensures the existence of a solution to equation (17), and as a result of equation (18) then

$$\Gamma^\mu = 0.$$

Thus, by a suitable choice of coordinates, the contracted Christoffel symbols $\Gamma^\mu$ can be locally made to vanish. This construction determines the coordinates uniquely.

Using the wave coordinates described in the previous section, equation (16) takes the form

$$g^{\lambda\rho}\partial_\lambda\partial_\rho g_{\mu\nu} - 2g_{\lambda\rho}g^{\sigma\tau}\Gamma^\lambda{}_{\sigma\mu}\Gamma^\rho{}_{\tau\nu} - 4\Gamma^\sigma{}_{\lambda\rho}g^{\lambda\tau}g_{\sigma(\mu}\Gamma^\rho{}_{\nu)\tau} = 0 \tag{19}$$

that is, one obtains a system of quasilinear wave equations for the components of the metric tensor $g_{\mu\nu}$. For this system, the local Cauchy problem with appropriate data is well-posed —one can show the existence and uniqueness of solutions and their stable dependence on the data —see e.g. [5]. This system of equations is called the reduced Einstein field equations. Similarly, the procedure leading to it is called a *hyperbolic reduction* of the Einstein vacuum equations. It is worth stressing the the relevance of obtaining a reduced version of the Einstein field equations in a manifestly hyperbolic form is that for these equations one readily has a developed theory of existence and uniqueness available. The introduction of a specific system of coordinates via the use of wave coordinates breaks the tensorial character of the Einstein field equations (4). Given a solution to the reduced Einstein field equations, the latter will also imply a solution to the actual equations as long as the coordinates $x^\mu$ satisfy equation (18) for the chosen coordinate source function appearing in the reduced equation. Thus, the standard procedure to prove local existence of solutions to the Einstein field equation with prescribed initial data is to show first the existence for a particular reduction of the equations and then prove, afterwards, that the coordinates that have been used satisfy the coordinate condition (18). This argument will be detailed in a subsequent section once other issues have been addressed.

The domain on which the coordinates $(x^\mu)$ form a good coordinate system depends on the initial data prescribed and the solution $g_{\mu\nu}$ itself. Since the information on $g_{\mu\nu}$ is only obtained by solving equation (18), there is little that can be said *a priori* about the domain of existence of the coordinates.

The data for the reduced equation (19) consists of a prescription of $g_{\mu\nu}$ and $\partial_\lambda g_{\mu\nu}$ at some initial time $t = 0$.

## 4.4 The propagation of the wave coordinates condition

To conclude the discussion it is now shown that under suitable conditions the reduced Einstein equations imply a solution of the actual Einstein field equations. This in fact, is equivalent to showing that if the contracted Christoffel symbols $\Gamma^\mu \equiv g^{\nu\lambda}\Gamma^\mu{}_{\nu\lambda}$ vanish initially, then they also vanish at any later time.

The starting point of this discussion is the observation that the *reduced Einstein field equations* can be written as

$$R_{\mu\nu} = \nabla_{(\mu}\Gamma_{\nu)}.$$

Now, using the contracted Bianchi identity

$$\nabla^\mu(R_{\mu\nu} - \frac{1}{2}Rg_\mu) = 0,$$

it follows that

$$\Box\Gamma_\mu + R^\nu{}_\mu Q_\mu = 0.$$

This is a wave equation for the contracted Christoffel symbol. In view of its homogeneity, if

$$\Gamma_\mu = 0, \qquad \nabla_\nu \Gamma_\mu = 0, \qquad \text{at } \ t = 0, \tag{20}$$

then $\Gamma_\mu = 0$ at later times and accordingly $R_{\mu\nu} = 0$. That is, one has a solution to the Einstein field equations.

# 5 The $3+1$ decomposition in General Relativity

In order to understand the structure of the initial value problem in General Relativity one has to do break the covariance of the theory and introduce a privileged time direction which, in turn, is used to decompose the equations of the theory.

## 5.1 Submanifolds of spacetime

Intuitively, a *submanifold* of $\mathcal{M}$, is a set $\mathcal{N} \subset \mathcal{M}$ which inherits a manifold structure from $\mathcal{M}$. The precise definition of a submanifold requires the concept of *embedding* —essentially a map $\varphi : \mathcal{N} \to \mathcal{M}$ which is injective and structure preserving; in particular the restriction $\varphi : \mathcal{N} \to \varphi(\mathcal{N})$ is a diffeomorphism. In terms of the above concepts, a submanifold $\mathcal{N}$ is the image $\varphi(\mathcal{N}) \subset \mathcal{M}$ of a $k$-dimensional manifold ($k < n$). Very often it is convenient to identify $\mathcal{N}$ with $\varphi(\mathcal{N})$.

In what follows we will mostly be concerned with 3-dimensional manifolds. It is customary to call these *hypersurfaces*. A generic hypersurfaces will be denoted by $\mathcal{S}$.

## 5.2 Foliations of spacetime

The presentation in this section follows very closely that of [2] Section 2.3. In what follows, we assume that the spacetime $(\mathcal{M}, g_{ab})$ is *globally hyperbolic*. That is, we assume that its topology is that of $\mathbb{R} \times \mathcal{S}$, where $\mathcal{S}$ is an orientable 3-dimensional manifold. A slightly different way of saying this is that the spacetime is assumed to be *foliated* by 3-manifolds (*hypersurfaces*) $\mathcal{S}_t$, $t \in \mathbb{R}$ such that

$$\mathcal{M} = \bigcup_{t\in\mathbb{R}} \mathcal{S}_t,$$

where we identify the leaves $\mathcal{S}_t$ with $\{t\} \times \mathcal{S}$. It is assumed that the hypersurfaces $\mathcal{S}_t$ do not intersect each other. It is customary to think of the hypersurface $\mathcal{S}_0$ as an initial hypersurface on which the initial information giving rise to the spacetime is to be prescribed. Globally hyperbolic spacetimes constitute the natural class of spacetime on which to pose an initial value problem for General Relativity.

In what follows it will be convenient to assume that the hypersurfaces $\mathcal{S}_t$ arise as the level surfaces of of a scalar function $t$ which will be interpreted as a *global time function*. From $t$ one can define the the covector

$$\omega_a = \nabla_a t.$$

By construction $\omega_a$ denotes the normal to the leaves $\mathcal{S}_t$ of the foliation. The covector $\omega_a$ is *closed* —that is,

$$\nabla_{[a}\omega_{b]} = \nabla_{[a}\nabla_{b]}t = 0.$$

From $\omega_a$ one defines a scalar $\alpha$ called the *lapse function* via

$$g^{ab}\nabla_a t \nabla_b t = \nabla^a t \nabla_a t \equiv -\frac{1}{\alpha^2}.$$

The lapse measures how much proper time elapses between neighbouring time slices along the direction given by the normal vector $\omega^a \equiv g^{ab}\omega_b$. In what follows, it is assumed that $\alpha > 0$. Notice that $\omega^a$ is assumed to be timelike so that the hypersurfaces $\mathcal{S}_t$ are spacelike.

In what follows we define the unit normal $n_a$ via

$$n_a \equiv -\alpha\omega_a.$$

The minus sign in the last definition is chosen so that $n^a$ points in the direction of increasing $t$. One can readily verify that $n^a n_a = -1$. One thinks of $n^a$ as the 4-velocity of a *normal* observer whose worldline is always orthogonal to the hypersurfaces $\mathcal{S}_t$.

The spacetime metric $g_{ab}$ induces a 3-dimensional Riemannian metric $h_{ij}$ on $\mathcal{S}_t$ —the indices $_{ij}$ are being used here to indicate that the induced metric is an intrinsically 3-dimensional object. The tensors $g_{ab}$ and $h_{ij}$ are related to each other via

$$h_{ab} \equiv g_{ab} + n_a n_b.$$

Note that although $h_{ij}$ is a 3-dimensional object, in the previous formula spacetime indices $_{ab}$ are used —i.e. we regard the 3-metric as an object living on spacetime. One can also use $h_{ab}$ to measure distances within $\mathcal{S}_t$. In order to see that $h_{ab}$ is purely spatial —i.e. it has no component along $n^a$— one contracts with the normal:

$$n^a h_{ab} = n^a g_{ab} + n_a n^a n_b = n_b - n_b = 0.$$

Intuition suggests that $h_{ab}$ calculates the same distance as $g_{ab}$ and then kills off the timelike contribution —i.e. the components along $n_a n_b$. The inverse 3-metric $h^{ab}$ is obtained by raising indices with

$$h^{ab} = g^{ab} + n^a n^b$$

The 3-metric $h_{ab}$ can be used as a *projector tensor* which projects all geometric objects lying along $n^a$. Effectively, $h_{ab}$ decomposes tensors into a *purely spatial part* which lies on the hypersurfaces $\mathcal{S}_t$ and a *timelike part* normal to the hypersurface. In actual computations it is convenient to consider

$$h_a{}^b = \delta_a{}^b + n_a n^b.$$

Given a tensor $T_{ab}$ its spatial part, to be denoted by $T^{\perp}_{ab}$, is defined to be

$$T^{\perp}_{ab} \equiv h_a{}^c h_b{}^d T_{cd}.$$

One can also define a *normal projector* $N_a{}^b$ as

$$N_a{}^b \equiv -n_a n^b = \delta_a{}^b - h_a{}^b.$$

In terms of these operators an arbitrary projector can be decomposed as

$$v^a = \delta^a{}_b v^b = (h_b{}^a + N_b{}^a)v^b = v^{\perp a} - n^a n_b v^b.$$

The 3-metric $h_{ij}$ defines in a unique manner a covariant derivative $D_i$ —the Levi-Civita connection of $h_{ij}$. As in the previous paragraphs it is convenient to make use of a 4-dimensional (spacetime) perspective so that we write $D_a$. Following this point of view one requires $D_a$ to be torsion-free and compatible with the metric $h_{ab}$. Taking this into account one defines for a scalar $\phi$

$$D_a\phi \equiv h_a{}^b \nabla_b \phi,$$

and, say, for a $(1,1)$-tensor

$$D_a T^b{}_c \equiv h_a{}^d h_e{}^b h_c{}^f \nabla_d T^e{}_f,$$

with an obvious extension to other tensors. In coordinates, the covariant derivative $D_a$ is associated to the *spatial Christoffel symbols*

$$\gamma^\mu{}_{\nu\lambda} = \frac{1}{2} h^{\mu\rho}(\partial_\nu h_{\rho\lambda} + \partial_\lambda h_{\nu\rho} - \partial_\rho h_{\nu\lambda}).$$

Being a covariant derivative, one can naturally associate a curvature tensor $r^a{}_{bcd}$ to $D_a$ by considering its commutator:

$$D_a D_b v^c - D_b D_a v^c = r^c{}_{dab} v^d$$

One can verify that $r^c{}_{dab} n^d = 0$. Similarly, one can define the Ricci tensors and scalar as

$$r_{db} \equiv r^c{}_{dcb}, \qquad r \equiv g^{ab} r_{ab}.$$

## 5.3 Extrinsic curvature

The Einstein field equation $R_{ab} = 0$ imposes some conditions on the 4-dimensional Riemann tensor $R^a{}_{bcd}$. Thus, in order to understand the implications of the Einstein equations one needs to decompose $R^a{}_{bcd}$ into spatial parts. This decomposition naturally involves $r^a{}_{bcd}$, but there is more to it as this last object is purely spatial and is computed directly from $h_{ab}$. Hence, $r^a{}_{bcd}$ measures the *intrinsic curvature* of the hypersurface $\mathcal{S}_t$. This tensor provides no information about how $\mathcal{S}_t$ fits in $(\mathcal{M}, g_{ab})$. The missing piece of information is contained in the so-called *extrinsic curvature.*

The extrinsic curvature is defined as the following projection of the spacetime covariant derivative of the normal to $\mathcal{S}_t$:

$$K_{ab} \equiv -h_a{}^c h_b{}^d \nabla_{(c} n_{d)} = -h_a{}^c h_b{}^d \nabla_c n_d.$$

The second equality follows from the fact that $n_a$ is *rotation free* —see the exercise sheet 2. By construction, the extrinsic curvature is symmetric and purely spatial. It measures how the normal to the hypersurface changes from point to point. As a consequence, the extrinsic curvature also measures the rate at which the hypersurface deforms as it is carried along the normal.

A related concept to extrinsic curvature is that of the *acceleration* of a foliation

$$a_a \equiv n^b \nabla_b n_a.$$

Using $n^d \nabla_c n_d = 0$, one can compute

$$\begin{aligned} K_{ab} &= -h_a{}^c h_b{}^d \nabla_c n_d = -(\delta_a{}^c + n_a n^c)(\delta_b{}^d + n_b n^d) \\ &= -(\delta_a{}^c + n_a n^c)\delta_b{}^d \nabla_c n_d = -\nabla_a n_b - n_a a_b. \end{aligned}$$

An alternative expression of the extrinsic curvature is given in terms of the Lie derivative. To obtain this one computes

$$\begin{aligned} \mathcal{L}_n h_{ab} &= \mathcal{L}_n (g_{ab} + n_a n_b) = 2\nabla_{(a} n_{b)} + n_a \mathcal{L}_n n_b + n_b \mathcal{L}_n n_a \\ &= 2(\nabla_{(a} n_{b)} + n_{(a} a_{b)}) = -2K_{ab}. \end{aligned}$$

A related object is the so-called *mean curvature*:

$$K \equiv g^{ab} K_{ab} = h^{ab} K_{ab}.$$

One can compute (exercise):

$$K = -\mathcal{L}_n (\ln \det h).$$

Thus the mean curvature measures the fractional change in 3-dimensional volume along the normal $n^a$.

## 5.4 The Gauss-Codazzi and Codazzi-Mainardi equations

Given the extrinsic curvature of an hypersurface $\mathcal{S}_t$, we now look how this relates to the curvature of spacetime. A computation using the definitions of the previous section shows that

$$D_a D_b v^c = h_a{}^p h_b{}^q h_r{}^c \nabla_p \nabla_q v^r - K_{ab} h_r{}^c n^p \nabla_p v^r - K_a{}^c K_{bp} v^p.$$

Combining with the commutator

$$D_a D_b v^c - D_b D_a v^c = r^c{}_{dab} v^d,$$

after some manipulations one obtains

$$r_{abcd} + K_{ac} K_{bd} - K_{ad} K_{cb} = h_a{}^p h_b{}^q h_c{}^r h_d{}^s R_{pqrs}. \tag{21}$$

This equation is called the *Gauss-Codazzi equation.* It relates the spatial projection of the spacetime curvature tensor to the 3-dimensional curvature.

A further important identity arises from considering projections of $R_{abcd}$ along the normal direction. This involves a spatial derivative of the extrinsic curvature. Namely,

$$D_a K_{bc} = h_a{}^p h_b{}^q h_c{}^r \nabla_p K_{qr}.$$

From this expression after some manipulations one can deduce

$$D_b K_{ac} - D_a K_{bc} = h_a{}^p h_b{}^q h_c{}^r n^s R_{pqrs}. \tag{22}$$

This equation is called the *Codazzi-Mainardi equation.*

In the sequel, we explore the consequences of equations (21) and (22) for the initial value problem in General Relativity.

### 5.5 The constraint equations of General Relativity

The $3+1$ decomposition of the Einstein field equations allows to identify the intrinsic metric and the extrinsic curvature of an initial hypersurface $\mathcal{S}_0$ as the initial data to be prescribed for the evolution equations of General Relativity.

In what follows we will make use of the Gauss-Codazzi and the Codazzi-Mainardi equations to extract the consequences of the vacuum Einstein field equations $R_{ab} = 0$ on a hypersurface $\mathcal{S}$.

Contracting the Gauss-Codazzi equation (21) one one find that

$$h^{pr} h_b{}^q h_d{}^s R_{pqrs} = r_{bd} + K K_{bd} - K^c{}_d K_{cb},$$

where $K \equiv h^{ab} K_{ab}$ denotes the trace of the extrinsic curvature. A further contraction yields

$$h^{pr} h^{qs} R_{pqrs} = r + K^2 - K_{ab} K^{ab}.$$

Now, the left-hand side can be expanded into

$$\begin{aligned} h^{pr} h^{qs} R_{pqrs} &= (g^{pr} + n^p n^s)(g^{qs} + n^q n^s) \\ &= R + 2n^p n^r R_{pr} + n^p n^r n^q n^s R_{pqrs} = 0. \end{aligned}$$

The last term vanishes because of the symmetries of the Riemann tensor. Combining the equations from the previous calculations one obtains the so-called *Hamiltonian constraint*:

$$r + K^2 - K_{ab} K^{ab} = 0. \tag{23}$$

One can proceed in a similar way with the Codazzi-Mainardi equation (22). Contracting once one has that

$$D^b K_{ab} - D_a K = h_a{}^p h^{qr} n^s R_{pqrs}.$$

The right hand side of this equation can be, in turn, expanded as

$$\begin{aligned} h_a{}^p h^{qr} n^s R_{pqrs} &= -h_a{}^p (g^{qr} + n^p n^r) n^s R_{qprs} \\ &= -h_a{}^p n^s R_{ps} - h_a{}^p n^q n^r n^s R_{pqrs} = 0, \end{aligned}$$

where in the last equality one makes use, again of the vacuum Equations and the symmetries of the Riemann tensor. Combining the previous expressions one obtains the so-called *momentum constraint*

$$D^b K_{ab} - D_a K = 0. \tag{24}$$

The Hamiltonian and momentum constraint equations (23) and (24) involve only the 3-dimensional intrinsic metric, the extrinsic curvature and their spatial derivatives. They are the conditions that allow a 3-dimensional slice with data $(h_{ab}, K_{ab})$ to be embedded in a 4-dimensional spacetime $(\mathcal{M}, g_{ab})$. The existence of the constraint equations implies that the data for the Einstein field equations cannot be prescribed freely. The nature of the constraint equations and possible procedures to solve them will be analysed later in these notes.

An important point still to be clarified is whether the fields $h_{ij}$ and $K_{ij}$ indeed correspond to data for the Einstein field equations. To see this, one has to analyse the evolution equations implied by the Einstein field equations.

### 5.5.1 The constraints for the Maxwell equations

The equations of other physical theories also imply constraint equations. The classical example in this respect is given by the Maxwell equations. To analyse these constraint equations it is convenient to introduce the *electric* and *magnetic parts* of the Faraday tensor $F_{ab}$:

$$E_a \equiv F_{ab}n^b, \qquad B_a \equiv \frac{1}{2}\epsilon_{ab}{}^{cd}F_{cd}n^b = F^*_{ab}n^b.$$

A calculation then shows that the Maxwell equations imply the constraint equations

$$D^a E_a = 0, \qquad D^a B_a = 0.$$

These constraints correspond to the well-known *Gauss laws for the electric and magnetic fields.* Thus, it follows that data for the Maxwell equations cannot be prescribed freely. The initial value of the electric and magnetic parts of the Faraday tensor must be divergence free.

**Remark.** Notice, by contrast that the wave equation for a scalar field $\phi$ implies no constraint equations. Thus, the data for this equation can be prescribed freely.

## 5.6 The evolution equations of General Relativity

To discuss the evolution equations of General Relativity one needs a further geometric identity —the so-called *Ricci equation.* To obtain this, one computes

$$\begin{aligned}\mathcal{L}_n K_{ab} &= n^c\nabla_c K_{ab} + 2K_{c(a}\nabla_{b)}n^c \\ &= -n^c\nabla_c\nabla_a n_b - n^c\nabla_c(n_a a_b) - 2K_{c(a}K_{b)}{}^c - 2K_{c(a}n_{b)}a^c.\end{aligned}$$

Now, making use of the commutator

$$\nabla_c\nabla_a n_b - \nabla_a\nabla_c n_b = R_{dbac}n^d,$$

one obtains

$$\mathcal{L}_n K_{ab} = -n^d n^c R_{dbac} - n^c\nabla_a\nabla_c n_b - n^c a_b\nabla_c n_a - n^c n_a\nabla_c a_b - 2K^c{}_{(a}K_{b)c} - 2K_{c(a}n_{b)}a^c.$$

Furthermore, using $a_b = n^c\nabla_c n_b$ and

$$n^c\nabla_a\nabla_c n_b = \nabla_a a_b - (\nabla_a n^c)(\nabla_c n_b) = \nabla_a a_b - K_a{}^c K_{cb} - n_a a^c K_{cb},$$

after some cancellations one gets

$$\mathcal{L}_n K_{ab} = -n^d n^c R_{dbac} - \nabla_a a_b - n^c n_a\nabla_c a_b - a_a a_b - K^c{}_b K_{ac} - K_{ca}n_b a^c. \tag{25}$$

It is observed that $\mathcal{L}_n K_{ab}$ is a spatial object in the sense that $n^a\mathcal{L}_n K_{ab} = 0$ (exercise!). This means that in equation (25) one can project the free indices to obtain

$$\mathcal{L}_n K_{ab} = -n^d n^c h_a{}^q h_b{}^r R_{drqc} - h_a{}^q h_b{}^r\nabla_q a_r - a_a a_b - K_b{}^c K_{ac}.$$

Finally, using the identity (exercise!)

$$D_a a_b = -a_a a_b + \frac{1}{\alpha}D_a D_b\alpha,$$

some simplifications yield the desired *Ricci equation:*

$$\mathcal{L}_a K_{ab} = n^d n^c h_a{}^q h_b{}^r R_{drcq} - \frac{1}{\alpha}D_a D_b\alpha - K_b{}^c K_{ac}. \tag{26}$$

Geometrically, this equation relates the derivative of the extrinsic curvature in the normal direction to an hypersurface $\mathcal{S}$ to a time projection of the the Riemann tensor.

The discussion from the previous paragraphs suggests that the Einstein field equations will imply an *evolution* of the data $(h_{ab}, K_{ab})$. Previously, it has been assumed that the spacetime $(\mathcal{M}, g_{ab})$ is foliated by a time function $t$ whose level surfaces correspond to the leaves of the foliation. Recalling that $\omega_a = \nabla_a t$, we consider now a vector $t^a$ (the *time vector*) such that

$$t^a = \alpha n^a + \beta^a, \qquad \beta_a n^a = 0. \tag{27}$$

The vector $\beta^a$ is called the *shift vector*. The time vector $t^a$ will be used to *propagate coordinates* from one time slice to another. In other words, $t^a$ connects points with the same spatial coordinate —hence, the shift vector measures the amount by which the spatial coordinates are shifted within a slice with respect to the normal vector. Together, the lapse and shift determine how coordinates evolve in time. The choice of these functions is fairly arbitrary and, hence, they are known as *gauge functions*. The lapse function reflects the freedom to choose the sequence of time slices, pushing them forward by different amounts of proper time at different spatial points on a slice —this idea is usually known as the *many-fingered nature of time*. The shift vector reflects the freedom to relabel spatial coordinates on each slices in an arbitrary way. Observers *at rest* relative to the slices follow the normal congruence $n^a$ and are called *Eulerian observers*, while observers following the congruence $t^a$ are called *coordinate observers*.

It is observed that as a consequence of expression (27) one has $t^a \nabla_a t = 1$ so that the integral curves of $t^a$ are naturally parametrised by $t$.

Recalling that $K_{ab} = -\frac{1}{2}\mathcal{L}_n h_{ab}$ and using equation (27) one concludes that

$$\mathcal{L}_t h_{ab} = -2\alpha K_{ab} + \mathcal{L}_\beta h_{ab}, \tag{28}$$

where it has been used that

$$\mathcal{L}_t h_{ab} = \mathcal{L}_{\alpha n + \beta} h_{ab} = \alpha \mathcal{L}_n h_{ab} + \mathcal{L}_\beta h_{ab}.$$

Equation (28) will be interpreted as an evolution equation for the intrinsic metric $h_{ab}$. To construct a similar equation for the extrinsic curvature one makes use of the Ricci equation (26). It is observed that

$$\begin{aligned} n^d n^c h_a{}^q h_b{}^r R_{drcq} &= h^{cd} h_a{}^q h_b{}^r R_{drcq} - h_a{}^q h_b{}^r R_{rq} \\ &= h^{cd} h_a{}^q h_b{}^r R_{drcq}, \end{aligned}$$

where to obtain the second equality the vacuum Einstein field equations $R_{ab} = 0$ have been used. The remaining term, $h^{cd} h_a{}^q h_b{}^r R_{drcq}$, is dealt with the Gauss-Codazzi equation (21). Finally, using that

$$\mathcal{L}_t K_{ab} = \mathcal{L}_{\alpha n + \beta} K_{ab} = \alpha \mathcal{L}_n K_{ab} + \mathcal{L}_\beta K_{ab},$$

one concludes that

$$\mathcal{L}_t K_{ab} = -D_a D_b \alpha + \alpha(r_{ab} - 2K_{ac}K^c{}_b + K K_{ab}) + \mathcal{L}_\beta K_{ab}. \tag{29}$$

This is the desired evolution equation for $K_{ab}$. Equations (28) and (29) are called the *ADM evolution equations*. They determine the evolution of the data $(h_{ab}, K_{ab})$. Together with the constraint equations (23) and (24) they are completely equivalent to the vacuum Einstein field equations.

**Remark.** The evolution equations (28) and (29) first order equations —compare with the wave equation for the components of the metric $g_{ab}$ discussed in Section 5.3. However, the equations are not hyperbolic! Thus, one cannot apply directly the standard PDE theory to assert existence of solutions. Nevertheless, there are some more complicated versions of them which do have the property of hyperbolicity.

### 5.6.1 The Maxwell evolution equations

As in the case of the constraint equations, it is instructive to look at the Maxwell equations to obtain some insight into the structure of the ADM evolution equations. Making use of the electric and magnetic part of the Faraday tensor, a computation of $\mathcal{L}_t E_a$ and $\mathcal{L}_t B_a$ together with the Maxwell equations allows to show that

$$\mathcal{L}_t E_a = \epsilon_{abc} D^b E^c + \mathcal{L}_\beta E_a,$$
$$\mathcal{L}_t B_a = -\epsilon_{abc} D^b B^c + \mathcal{L}_\beta B_a.$$

Observe the similarity with the ADM equations!

## 5.7 The $3+1$ form of the spacetime metric

The discussion of the evolution equations given in the previous section has been completely general. By this we mean that the only assumption that has been made about the spacetime is that it is globally hyperbolic so that a foliation and a corresponding time vector exist. The discussion of the $3+1$ decomposition can be further particularised by introducing adapted coordinates. In this section we briefly discuss how this can be done.

Firstly, it is recalled that the hypersurfaces of the foliation of a spacetime $(\mathcal{M}, g_{ab})$ can be given as the level surfaces of a time function $t$. Now, we already have seen that $\nabla_a t^a = 1$. The latter combined with $\nabla_a t = (1,0,0,0)$ readily imply that

$$t^\mu = (1,0,0,0).$$

The latter implies, that the Lie derivative along the direction of $t^a$ is simply a partial derivative —that is,

$$\mathcal{L}_t = \partial_t.$$

Clearly, from the previous discussion it also follows that the spatial components of the unit normal must vanish —i.e. one has that $n_i = 0$. Since the contraction of spatial vectors with the normal must vanish, it follows that all components of spatial tensors with a contravariant index equal to zero must vanish. For the shift vector one has that $n_a\beta^a = n_0\beta^0 = 0$ so that

$$\beta^\mu = (0, \beta^\gamma).$$

Since one has that $t^a = \alpha n^a + \beta^a$, it follows then that

$$n^\mu = (\alpha^{-1}, -\alpha^{-1}\beta^\gamma).$$

Moreover, from the normalisation condition $n_a n^a = -1$ one finds

$$n_\mu = (-\alpha, 0, 0, 0).$$

Now, recalling that $h_{ab} = g_{ab} + n_a n_b$ one concludes that

$$h_{\alpha\beta} = g_{\alpha\beta}.$$

In these *adapted coordinates* the 3-metric of the hypersurfaces of the foliation are simply the spatial part of the spacetime metric $g_{ab}$. Moreover, since the time components of spatial contravariant tensors have to vanish, one also has that $h^{\mu 0} = 0$. One concludes that one can write

$$g^{\mu\nu} = h^{\mu\nu} - n^\mu n^\nu = \begin{pmatrix} -\alpha^{-2} & \alpha^{-2}\beta^\gamma \\ \alpha^{-2}\beta^\delta & h^{\gamma\delta} - \alpha^{-2}\beta^\gamma\beta^\delta \end{pmatrix}.$$

This last expression can be inverted to yield

$$g_{\mu\nu} = \begin{pmatrix} -\alpha^2 + \beta_\gamma\beta^\gamma & \beta_\gamma \\ \beta_\delta & h_{\gamma\delta} \end{pmatrix},$$

where $\beta_\gamma \equiv h_{\gamma\delta}\beta^\delta$. Alternatively, one has that

$$g = -\alpha^2 \mathrm{d}t^2 + h_{\gamma\delta}(\beta^\gamma \mathrm{d}t + \mathrm{d}x^\gamma)(\beta^\delta \mathrm{d}t + \mathrm{d}x^\delta).$$

The latter is known as the $3+1$ *form* of the spacetime metric.

### 5.7.1 Summary of the $3+1$ decomposition of the Einstein vacuum equations

The adapted coordinates used in the previous paragraphs can be used to simplify the presentation of the constraint and evolution equations. It readily follows that these equations can be written as

$$
\begin{aligned}
&r + K^2 - K_{ij}K^{ij} = 0,\\
&D^j K_{ij} - D_j K = 0,\\
&\partial_t h_{ij} = -2\alpha K_{ij} + D_i\beta_j + D_j\beta_i,\\
&\partial_t K_{ij} = -D_i D_j \alpha + \alpha(r_{ij} - 2K_{ik}K^k{}_j + KK_{ij}) + \beta^k D_k K_{ij} + K_{ik}D_j\beta^k + K_{kj}D_i\beta^k.
\end{aligned}
$$

### 5.7.2 An example: the Schwarzschild spacetime

As we have already discussed, the metric Schwarzschild spacetime can be expressed in standard coordinates in terms of the line element

$$
g = -\left(1 - \frac{2m}{r}\right)\mathrm{d}t^2 + \left(1 - \frac{2m}{r}\right)^{-1}\mathrm{d}r^2 + r^2\mathrm{d}\theta^2 + r^2\sin^2\theta\mathrm{d}\varphi^2.
$$

This form of the metric is not the best one for a $3+1$ decomposition of the spacetime. Instead, it is better to introduce an *isotropic radial coordinate* $\bar{r}$ via

$$
r = \bar{r}\left(1 + \frac{m}{2\bar{r}}\right)^2.
$$

In terms of the later one obtains the line element of the Schwarzschild spacetime in *isotropic coordinates*:

$$
g = -\left(\frac{1 - \frac{m}{2\bar{r}}}{1 + \frac{m}{2\bar{r}}}\right)^2 \mathrm{d}t^2 + \left(1 + \frac{m}{2\bar{r}}\right)^4 (\mathrm{d}\bar{r}^2 + \bar{r}^2\mathrm{d}\theta^2 + \bar{r}^2\sin^2\theta\mathrm{d}\varphi).
$$

The normal $\omega_a = \nabla_a t$ is then readily given by

$$
\omega_\mu = (1, 0, 0, 0).
$$

Thus, one readily reads the lapse function to be

$$
\alpha = \frac{1 - \frac{m}{2\bar{r}}}{1 + \frac{m}{2\bar{r}}},
$$

while the unit normal is

$$
n^\mu = \frac{1 + \frac{m}{2\bar{r}}}{1 - \frac{m}{2\bar{r}}}(1, 0, 0, 0).
$$

The spatial metric is then

$$
h = \left(1 + \frac{m}{2\bar{r}}\right)^4 (\mathrm{d}\bar{r}^2 + \bar{r}^2\mathrm{d}\theta^2 + \bar{r}^2\sin^2\theta\mathrm{d}\varphi).
$$

One also notices that the shift vanishes —i.e. one has that

$$
\beta^i = 0.
$$

Since $\beta^\alpha = 0$ and $h_{\alpha\beta}$ is independent of time, one can readily finds that the extrinsic curvature vanishes

$$
K_{\alpha\beta} = 0.
$$

The isotropic form of the Schwarzschild metric yields a foliation of spacetime that follows the static symmetry of the spacetime. In this foliation, the intrinsic 3-metric of the leaves does not seems to evolve. Any other foliation not aligned with the static Killing vector will give rise to a non-trivial evolution for both $h_{\alpha\beta}$ and $K_{\alpha\beta}$.

# 6 A closer look at the constraint equations

The purpose of this section is to explore some aspects of the constraint equations, and in particular, the manner one could expect to solve them.

As shown in Section 6.5, the Einstein field equations imply the following constraint equations on a (spatial) hypersurface $\mathcal{S}$:

$$r + K^2 - K_{ij}K^{ij} = 0, \qquad \text{Hamiltonian constraint}$$
$$D^i K_{ij} - D_j K = 0. \qquad \text{Momentum constraint}$$

As already discussed, these equations constrain the possible choices of pairs $(h_{ij}, K_{ij})$ corresponding to initial data to the Einstein field equations. The constraint equations are intrinsic equations, that is, they only involve objects which are defined on the hypersurface $\mathcal{S}$ without any further reference to the "bulk" of the spacetime $(\mathcal{M}, g_{ab})$.

The Einstein constraint constitute a highly coupled, highly non-linear system of equations for $(h_{ij}, K_{ij})$. However, the main difficulty in constructing an solution to the equations lies in the fact that the equations are an underdetermined system: one has 4 equations for 12 unknowns —the independent components of two symmetric spatial tensors. Even exploiting the coordinate freedom to "kill off" three components of the tensors, one is still left with 9 unknowns. This feature indicates that there should be some freedom in the specification of *data* for the equations. The task is to identify what this free data is.

To render the problem manageable, we make a standard simplifying assumption and consider initial data sets for which $K_{ij} = 0$ everywhere on $\mathcal{S}$. This class of initial data are called *time symmetric*. The reason for this name is that if $K_{ij} = 0$ at $\mathcal{S}$ then the evolution equations imply that

$$\partial_t h_{ij} = 0, \qquad \text{on} \qquad \mathcal{S}.$$

This equation is invariant under the replacement $t \mapsto -t$. It follows that the resulting spacetime has a reflection symmetry with respect to the hypersurface $\mathcal{S}$ which can be regarded as a *moment of time symmetry.*

If $K_{ij} = 0$ everywhere on $\mathcal{S}$ then the momentum constraint is automatically solved, and the Hamiltonian constraint reduces to

$$r = 0.$$

That is, the initial 3-metric has to be such that its Ricci scalar vanishes —notice that this does not mean that the hypersurface is flat! Still, the time symmetric Hamiltonian constraint, regarded as an equation for $h_{ij}$, is highly non-linear. Moreover, one still has six unknowns and one equation —even choosing coordinates, one still is left with three unknowns . Now, clearly for an arbitrary metric $\bar{h}_{ij}$ one has that $\bar{r} \neq 0$. An idea to solve the constraint is then to introduce a factor that compensates this. This idea leads naturally to the notion of conformal transformations. Two metrics $h_{ij}$, $\bar{h}_{ij}$ are said to be conformally related if there exists a positive scalar $\vartheta$ (the *conformal factor*) such that

$$h_{ij} = \vartheta^4 \bar{h}_{ij}. \tag{30}$$

The power of $\vartheta$ used in the above equation is conventional and leads to simple equations in 3-dimensions. When discussing conformal transformations on dimensions $n \geq 4$, other powers may be more useful. In what follows, the metric $\bar{h}$ will be called the *background* metric. Loosely speaking, the conformal factor absorbs the overall scale of the metric. In the way presented, the conformal transformation introduced above is a mathematical trick to solve equations. At a deeper level, the conformal transformation defines an equivalence class of manifolds and metrics.

Given the conformal transformation (30), it is important to analyse its effects on other geometrical objects. In particular, recall that the 3-dimensional Christoffel symbols are given by

$$\Gamma^\alpha{}_{\beta\gamma} = \frac{1}{2} h^{\alpha\delta}(\partial_\beta h_{\gamma\delta} + \partial_\gamma h_{\beta\delta} - \partial_\delta h_{\beta\gamma}).$$

Substituting (30) into the previous equation one finds after some calculations that

$$\Gamma^\alpha{}_{\beta\gamma} = \bar{\Gamma}^\alpha{}_{\beta\gamma} + 2(\delta_\beta{}^\alpha \partial_\gamma \ln\vartheta + \delta_\gamma{}^\alpha \partial_\beta \ln\vartheta - \bar{h}_{\beta\gamma}\bar{h}^{\alpha\delta}\partial_\delta \ln\vartheta),$$

where $\bar{\Gamma}^\alpha{}_{\beta\gamma}$ denote the Christoffel symbols for the metric coefficients $\bar{h}_{\alpha\beta}$ and it has been used that $h^{ij} = \vartheta^{-4}\bar{h}^{ij}$ (exercise). A lengthier computation yields the following transformation law for the 3-dimensional Ricci tensor:

$$r_{ij} = \bar{r}_{ij} - 2(\bar{D}_i\bar{D}_j \ln\vartheta + \bar{h}_{ij}\bar{h}^{lm}\bar{D}_l\bar{D}_m \ln\vartheta) + 4(\bar{D}_i \ln\vartheta \bar{D}_j \ln\vartheta - \bar{h}_{ij}\bar{h}^{lm}\bar{D}_l \ln\vartheta \bar{D}_m \ln\vartheta).$$

Furthermore (and more importantly for our purposes) one has that

$$r = \vartheta^{-4}\bar{r} - 8\bar{\theta}^{-5}\bar{D}_k\bar{D}^k\vartheta.$$

In the above expressions $\bar{D}$ denotes the covariant derivative of the background metric $\bar{h}_{ij}$.

Using $r = 0$ in the transformation law for the Ricci scalar given above, one readily finds that

$$\bar{D}_k\bar{D}^k\vartheta - \tfrac{1}{8}\bar{r}\vartheta = 0. \tag{31}$$

In Differential Geometry, this equation is sometimes called the *Yamabe equation*. Given a fixed background metric $\bar{h}_{ij}$, equation (31) can be read as a differential condition for the conformal factor $\vartheta$. Given a solution $\vartheta$, one has by construction that $h_{ij} = \vartheta^4\bar{h}_j$ is such that $r = 0$ and one has constructed a solution to the time symmetric Einstein constraints. The Yamabe equation is *elliptic*: the operator $\bar{D}_k\bar{D}^k$ is the *Laplacian operator* associated to the metric $\bar{h}_{ij}$ —if $\bar{h}_{ij} = \delta_{ij}$ the flat metric in Cartesian coordinates, then

$$\bar{D}_k\bar{D}^k = \delta^{\alpha\beta}\partial_\alpha\partial_\beta = \partial_x^2 + \partial_y^2 + \partial_z^2.$$

Given a linear second order elliptic equation like (31), appropriate boundary conditions ensure the existence of a unique solution on $\mathcal{S}$.

Following the discussion of the previous paragraph, choose the flat metric as background metric. That is, let

$$\bar{h}_{\alpha\beta} = \delta_{\alpha\beta}.$$

In this case, the metric $h_{\alpha\beta} = \vartheta^4\delta_{\alpha\beta}$ is said to be *conformally flat*. Conformal flatness is an interesting property that Riemannian manifolds can possess. An important result is that conformal flatness is characterised locally by the vanishing of the *Cotton* tensor

$$b_{ijk} \equiv D_{[j}r_{k]i} - \tfrac{1}{4}h_{i[j}D_{k]}r.$$

For example, any spherically symmetric metric can be shown to be conformally flat. The purpose of assuming conformal flatness in our discussion is to provide a simplified setting to carry out calculations. In particular, one has that $\bar{r} = 0$ so that the Yamabe equation reduces to the *flat Laplace equation*

$$\bar{D}_k\bar{D}^k\vartheta = 0.$$

In the discussion of isolated systems (i.e. astrophysical sources) one is interested in solutions which are *asymptotically flat*. That is,

$$\vartheta = 1 + O(r^{-1}), \qquad \text{for} \qquad r \to \infty,$$

where $r^2 = x^2 + y^2 + z^2$ is the standard radial coordinate. Solutions to the Laplace equation with the above asymptotic behaviour are well known. In particular, a *spherically symmetric* solution is given by

$$\vartheta = 1 + \frac{m}{2r},$$

where $m$ is a constant. This solution to the time symmetric constraints is the 3-metric of the Schwarzschild spacetime in isotropic coordinates:

$$h = \left(1 + \frac{m}{2r}\right)^4 (\mathrm{d}r^2 + r^2\mathrm{d}\theta^2 + r^2\sin^2\theta\mathrm{d}\varphi^2).$$

The above 3-metric is singular at $r = 0$. This singularity, is a coordinate singularity. By considering the coordinate inversion

$$r = \frac{m^2}{4}\frac{1}{\bar{r}},$$

it can be seen that the metric transforms into

$$h = \left(1 + \frac{m}{2\bar{r}}\right)^4 (\mathrm{d}\bar{r}^2 + \bar{r}^2\mathrm{d}\theta^2 + \bar{r}^2\sin^2\theta\mathrm{d}\varphi^2).$$

The inversion transforms the metric into itself —that is, it is a discrete isometry. In particular, one has that the point $r = 0$ is can be mapped to infinity. Thus, the metric is perfectly regular everywhere and $r = 0$ is, in fact, the infinity of an asymptotically flat region. The hypersurface $\mathcal{S}$ has a non-trivial topology —it corresponds to a *wormhole*, see Figure 1. The radius given by $r = m/2$ corresponds to the minimum areal radius —this is called the *throat* of the black hole. Observe that $r = m/2 = \bar{m}/2$. The throat corresponds to the intersection of the black hole horizon with the hypersurface $\mathcal{S}$. The inversion reflects points with respect to the throat.

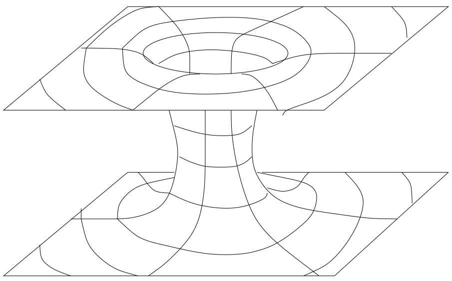

Figure 1: Embedding diagram of time-symmetric Schwarzschild data

The construction described in the previous paragraphs can be extended to include an arbitrary number of black holes. This is made possible by the linearity of the flat Laplace equation. Indeed, the conformal factor

$$\vartheta = 1 + \frac{m_1}{2r_1} + \frac{m_2}{2r_2} \tag{32}$$

with

$$r_1 = |x^i - x_1^i|, \qquad r_2 = |x^i - x_2^i|,$$

and where $x_1^i$ and $x_2^i$ denote the (fixed) location of two black holes with *bare masses* $m_1$ and $m_2$. The solution to the constraint equations given by the conformal factor (32) is called the *Brill-Lindquist solution* [3]. It describes a pair of black holes instantaneously at rest at a moment of time symmetry. This solution is used as initial data to simulate the head-on collision of two black holes. In this case what one finds is that each throat connects to is own asymptotically flat region. The drawing of the corresponding 3-dimensional manifold gives 3 different sheets, each corresponding to a different asymptotically flat region —see Figure 2.

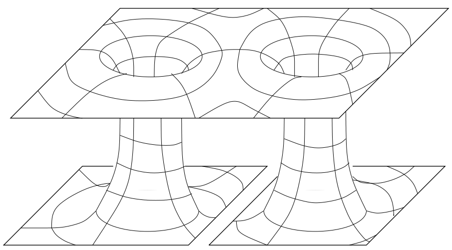

Figure 2: Embedding diagram of time-symmetric Brill-Lindquist data.

The flat Laplace equation can also be solved using the so-called *method of images* to obtain a solution with two holes and two asymptotic regions as depicted in the figure. This solution is known as *Misner data* [7]. This solution has a reflection symmetry through the throats, and has only two (as opposed to three of the Brill-Lindquist solution) asymptotically flat regions —see Figure 3. The solution is much more complicated than the Brill-Lindquist one, but still can be written explicitly.

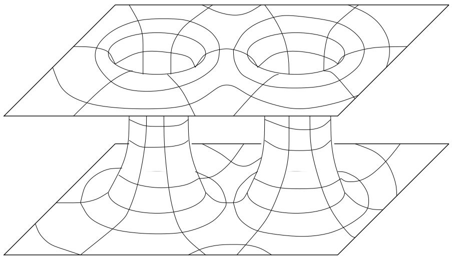

Figure 3: Embedding diagram of time-symmetric Misner data.

More complicated solutions to the constraint equations can be obtained by including a non-vanishing extrinsic curvature. In this way one can provide data for a rotating black hole or even a pair of rotating black holes spiralling around each other. The constraint equations in these cases have to be solved numerically.

# 7 Time independent solutions to the Einstein field equations

A systematic analysis of solutions to the vacuum Einstein field equations starts by considering time independent solutions. These solutions are interpreted as describing the gravitational field in the exterior of isolated bodies at rest or in uniform rotation in an otherwise empty Universe. The simplest case of a time independent solution is the Minkowski metric. More sophisticated examples are the Schwarzschild and Kerr spacetimes. The relevance of these two solutions is that they are thought to describe the end state of black hole evolution.

## 7.1 The time independent wave equation

Before analysing the Einstein field equations, it is useful to look at simpler toy models. To this end, consider a scalar field on the Minkowski spacetime satisfying the wave equation

$$(\Delta - \partial_t^2)\phi = 0,$$

where $\Delta$ denote the flat Laplacian. Now, consider time independent solutions —i.e. $\partial_t\phi = 0$. It follows that

$$\Delta\phi = 0.$$

The obvious observation is that an equation which is originally hyperbolic becomes elliptic under the assumption of time independence. This is a generic feature that can be observed in other theories —like the Maxwell equations and the Einstein field equations.

The energy of the scalar field at some time $t$ is given by

$$E(t) = \int_{\mathcal{S}_t} \left((\partial_t\phi)^2 + |\nabla\phi|^2\right) \mathrm{d}^3x.$$

In order to have finiteness of the energy one needs the boundary conditions

$$\phi(t, x^i),\ \partial_t\phi(t, x^i) \to \infty, \qquad \text{as} \qquad |x| \to \infty.$$

An important difference between hyperbolic equations and elliptic ones is that while in the former, properties of solutions can be localised and perturbations have finite propagation speed, for the latter the properties of solutions are global. For example, if $\phi = O(1/r)$ as $r \to \infty$ and $\Delta\phi = 0$, then it follows that $\phi = 0$. This follows from the integral

$$0 = \int_{\mathbb{R}^3} \phi \Delta \phi \mathrm{d}x^3 = \int_{\mathbb{R}^3} |\nabla \phi|^2 \mathrm{d}x^3,$$

where the Green's identity has been used. It follows that $|\nabla\phi|^2 = 0$ everywhere on $\mathbb{R}^3$ so that $\phi$ is constant. Due to the decay conditions, $\phi$ must necessarily vanish. This type of argument will be used repeatedly for the Einstein equations. In order to avoid the vanishing of $\phi$ in this case, one needs to consider the inhomogeneous problem —that is, one needs to consider sources.

## 7.2 Time independence: stationarity and staticity

Mathematically speaking, time independence is imposed by requiring on the spacetime $(\mathcal{M}, g_{ab})$ the existence of a timelike Killing vector $\xi^a$ —the spacetime is then said to be *stationary*. If, in addition, the Killing vector is hypersurface orthogonal —i.e. it is the gradient of some scalar function— then one says that $\xi^a$ is a *static Killing vector*. The Schwarzschild and Kerr solutions are, respectively, static and stationary. Stationary solutions to the Einstein field equations allow for the possibility of rotating gravitational fields.

Let $n_a$ denote the unit normal of an hypersurface $\mathcal{S}$. Now, if $\xi^a n_a = 0$, i.e. the Killing vector if orthogonal to $\mathcal{S}$, then a calculation shows that (exercise!) that

$$\xi_{[a} \nabla_b \xi_{c]} = 0. \tag{33}$$

The latter condition characterises *hypersurface orthogonality* —that is, a Killing vector is hypersurface orthogonal if and only if (33) holds. The proof of this result is classical and can be found in various textbooks —e.g. [12].

In a static spacetime, it is natural to choose adapted coordinates such that $\xi^\mu \partial_\mu = \partial_t$ —that is, the time coordinate is adapted to the flow lines of the Killing vector. Now, using the Killing vector condition $\mathcal{L}_\xi g_{ab} = 0$ and the definitions of $h_{\alpha\beta}$ and $K_{\alpha\beta}$ one can show that (exercise)

$$\partial_t h_{\alpha\beta} = \partial_t K_{\alpha\beta} = 0.$$

Now, recall the $3+1$ decomposition of the spacetime metric:

$$g = -\alpha^2 \mathrm{d}t^2 + h_{\alpha\beta}(\beta^\alpha \mathrm{d}t + \mathrm{d}x^\alpha)(\beta^\beta \mathrm{d}t + \mathrm{d}x^\beta).$$

In what follows we will analyse the simplifications introduced in the above line element by the assumption of staticity and the use of an adapted time coordinate. If the Killing vector is hypersurface orthogonal then the Killing vector has to be proportional to the normal to the hypersurface $\mathcal{S}$. That is, one has that

$$\xi_\mu = \alpha \nabla_\mu t.$$

However, the Killing vector can be decomposed in a lapse and a shift part:

$$\xi^a = N n^a + \beta^a.$$

Comparing both expressions finds that

$$\beta^\alpha = 0.$$

Thus, one has that

$$g = -\alpha^2 \mathrm{d}t^2 + h_{\alpha\beta} \mathrm{d}x^\alpha \mathrm{d}x^\beta,$$

with $h_{\alpha\beta}$ time independent. The time evolution equation for $h_{\alpha\beta}$ then takes the form

$$\partial_t h_{\alpha\beta} = -2\alpha K_{\alpha\beta} = 0.$$

Now, as the lapse cannot vanish one has that

$$K_{\alpha\beta} = 0.$$

That is, the hypersurfaces of the foliation adapted to the static Killing vector have no extrinsic curvature —this property is preserved as, already seen, $\partial_t K_{\alpha\beta} = 0$.

It follows from the discussion in the previous paragraphs that vacuum static solutions to the Einstein equations are characterised solely in terms of the lapse $\alpha$ and the 3-metric $h_{ij}$. In order to obtain equations for these quantities one considers the Hamiltonian constraint, equation (23), and the evolution equation for $K_{ij}$, equation (29). Setting $K_{\alpha\beta} = \partial_t K_{\alpha\beta} = 0$ readily yields

$$D_i D_j \alpha = r_{ij}, \tag{34a}$$

$$r = 0, \tag{34b}$$

where, as before, $r$ denotes the Ricci tensor of the 3-metric $h_{ij}$. These equations are known as the *static vacuum Einstein equations.*

## 7.3 Exploring the static equations

As a first example of the content and implications of the static equations (34a)-(34b), let $\mathcal{S} \approx \mathbb{R}^3$ —i.e. the hypersurface $\mathcal{S}$ has the topology of Euclidean space. Suppose that the fields $\alpha$ and $h_{ij}$ decay at infinity in such a way that

$$\alpha \to 1, \qquad h_{\alpha\beta} - \delta_{\alpha\beta} \to 0, \qquad \text{as} \qquad |x| \to \infty.$$

The first condition essentially means that it is assumed that the Killing vector behaves asymptotically like the static Killing vector of Minkowski spacetime. The second condition means that the 3-metic is assumed to be asymptotically flat (Euclidean) at infinity. Taking the trace of equation (34a) and using equation (34b) it follows that

$$\Delta\alpha = D_k D^k \alpha = 0.$$

Now, consider

$$0 = \int_{\mathcal{S}} \alpha \Delta \alpha \mathrm{d}^3 x = \int_{\mathcal{S}} |D\alpha|^2 \mathrm{d}^3 x,$$

again, as a consequence of the Gauss theorem. Thus

$$|D\alpha|^2 = h^{ij} D_i \alpha D_j \alpha = 0.$$

Hence $\alpha$ is a constant. Using the asymptotic condition $\alpha \to 1$ it follows $\alpha = 1$ everywhere. Using equation (34a) one concludes that

$$r_{ij} = 0.$$

Now, in 3-dimensions the Ricci tensor determines fully the curvature of the manifold. Thus one concludes

$$r_{ijkl} = 0,$$

so that is, $h_{\alpha\beta} = \delta_{\alpha\beta}$ —the Euclidean flat metric. The line element we have obtained is then

$$g = -\mathrm{d}t^2 + \delta_{\alpha\beta} \mathrm{d}x^\alpha \mathrm{d}x^\beta.$$

This solution is the Minkowski spacetime! This result is known as *Licnerowicz's theorem*:

**Theorem 1.** *The only globally regular static solution to the Einstein equations with $\mathcal{S}$ having trivial topology (i.e. $\mathcal{S} \approx \mathbb{R}^3$) and such that*

$$\alpha \to 1, \qquad h_{\alpha\beta} - \delta_{\alpha\beta} \to 0, \qquad \textit{as} \qquad |x| \to \infty$$

*is the Minkowski spacetime.*

The above theorem demonstrates the rigidity of the Einstein field equations. In order to obtain more interesting regular solutions, one requires either some matter sources or a non-trivial topology for $\mathcal{S}$ as in the case of the Schwarzschild spacetime —recall the Einstein-Rosen bridge! The result can be interpreted as a first, very basic uniqueness black hole result. If one wants to have a black hole solution one needs non-trivial topology!

### 7.4 Further results concerning static spacetimes

An important question in the analysis of static spacetimes is to characterise their asymptotic behaviour beyond the prescribed boundary conditions. Can one say more? The answer is contained in the following:

**Theorem 2** (Beig, 1980). *Every static vacuum solution to the Einstein equations satisfying*

$$\alpha \to 1, \qquad h_{\alpha\beta} - \delta_{\alpha\beta} \to 0, \qquad as \qquad |x| \to \infty$$

*is Schwarzschildean to leading order in $1/r$. That is,*

$$\alpha^2 = 1 - \frac{2m}{r} + O(1/r^2), \qquad h_{\alpha\beta} - \delta_{\alpha\beta} = \frac{2m}{r}\delta_{\alpha\beta} + O(1/r^2).$$

The proof of this result is already quite involved and, hence, it will not be discussed. Observe that in the previous result the regularity of $\mathcal{S}$ is not required. Also, there could be bounded sources somewhere in the interior. The lapse $\alpha$ can be interpreted as relativistic generalisation of a Newtonian potential. The theorem can be improved to include higher order multipoles. These lead to a *multipolar expansion* of the gravitational field. These multipoles characterise in a unique manner static solutions:

**Theorem 3** (Beig & Simon, 1981; Friedrich 2006). *Given an asymptotically flat static solution to the Einstein vacuum equations, one obtains a unique sequence of multipole moments. Conversely, given a sequence of multipole moments, if the lapse constructed from this sequence is well defined, there exists a unique static spacetime associated to these multipoles.*

# 8 Energy and momentum in General Relativity

It is a well known feature of General Relativity that energy and momentum of the gravitational field cannot be localised. This is a direct consequence of the *equivalence principle*. Thus, one cannot define, for example, a density of energy for the gravitational field. However, it is still possible to define some global conserved quantities which, in turn, can be interpreted as the total energy of a gravitating system. These quantities behave in a similar way to electromagnetic charges —that is, they take the form of volume integrals which are transformed, in turn, into surface integrals.

In what follows let $(\mathcal{S}, h_{ij}, K_{ij})$ denote an initial data set for the vacuum Einstein field equations —i.e. they satisfy the constraints. Let $x^\alpha$ denote *asymptotically Cartesian coordinates* —i.e. a system of coordinate for which $h_{\alpha\beta}$ agrees with $\delta_{\alpha\beta}$ to first order. One defines the *ADM energy* as the surface integral

$$E = \frac{1}{16\pi}\int_{\mathcal{S}_\infty} (\partial^\beta h_{\alpha\beta} - \partial_\alpha h) n^\alpha \mathrm{d}S, \qquad h \equiv h_{\alpha\beta}\delta^{\alpha\beta}.$$

where $S_\infty$ denotes the *sphere at infinity*, and $n^\alpha$ is the outward pointing normal to the sphere. Similarly, the *ADM momentum* is given by

$$p^\alpha = \frac{1}{8\pi}\int_{\mathcal{S}_\infty} (K^\alpha{}_\beta - K\delta^\alpha{}_\beta) n^\alpha \mathrm{d}S.$$

Although it is not directly evident, these expressions can be shown to be coordinate independent quantities [1]. In particular, a change to another asymptotically Cartesian system gives the same ADM mass and momentum. The energy $E$ and the momentum $p^\alpha$ are the components of a 4-dimensional vector (4-vector) —the *ADM 4-momentum vector*:

$$p^\mu = (E, p^\alpha).$$

Another non-trivial observation is that if one has an initial data set $(\mathcal{S}, h_{ij}, K_{ij})$ satisfying

$$h_{\alpha\beta} - \delta_{\alpha\beta} = O(1/r), \qquad K_{ij} = O(1/r^2),$$

then one can readily verify that

$$E < \infty, \qquad p^\alpha < \infty.$$

The verification of the above statement for $p^\alpha$ makes use of the constraint equations.

To obtain intuition into the content of the ADM energy and momentum, it is convenient to evaluate them on the Schwarzschild spacetime. As before, we make use of the time symmetric hypersurface given in standard coordinates by a constant value of $t$. As already seen, for this hypersurface it has been seen that $K_{ij} = 0$. Moreover, one has that

$$h_{\alpha\beta} = \left(1 + \frac{m}{2r}\right)^4 \delta_{\alpha\beta}.$$

A calculation then shows that

$$E = m, \qquad p^\alpha = 0.$$

That is, the ADM energy of the time symmetric slice of the Schwarzschild spacetime coincides with its mass parameter.

## 8.1 Conservation of the ADM 4-momentum

As $p^\alpha$ provides a measure of the total energy of a gravitating system, it is natural to expect that its components satisfy some sort of conservation behaviour. Moreover, it is also to be expected that the components of $p^\mu$ transform as a 4-vector under Lorentz transformations.

For the first point mentioned in the previous paragraph, consider an evolution off the hypersurface $\mathcal{S}$ such that

$$\alpha = 1 + O(1/r), \qquad \beta^\alpha = O(1/r).$$

The latter corresponds to an evolution into nearby hypersurfaces $\mathcal{S}$ which are essentially a time translation at infinity. From the above assumptions it follows that

$$\mathcal{L}_\beta g_{\mu\nu} = O(1/r^4).$$

One then computes $\partial_t E$ to obtain

$$\partial_t E = \int_{\mathcal{S}_\infty} (\partial_t \partial^\beta h_{\alpha\beta} - \partial_t \partial_\alpha h) n^\alpha \mathrm{d}S.$$

Using the ADM evolution equations one can readily verify by inspection that

$$\partial_t \partial^\beta h_{\alpha\beta} - \partial_t \partial_\alpha h = O(1/r^3).$$

It follows then that

$$\partial_t E = 0.$$

A similar argument shows that $\partial_t p^\alpha = 0$. Thus, indeed, the components of $p^\mu$ are conserved, at least for evolutions which behave as a time translation at infinity.

## 8.2 Positivity of the energy

On intuitive grounds one would expect the ADM 4-momentum to satisfy some positivity properties. That this is the case is not at all obvious from the definitions in terms of surface integrals of the ADM energy and momentum given in the previous paragraphs.

In order to gain intuition into this question it is convenient to analyse a model problem. In this case we consider Newtonian gravity. Accordingly, let $\phi$ denote the gravitational potential and let $\rho$ denote the density of matter. In physically realistic situations one expects $\rho$ to be a function of bounded support —that is, it vanishes outside a compact set. This requirement fits naturally with the notion of an isolated system. The gravitational potential is related to the density via the Poisson equation

$$\Delta\phi = 4\pi G\rho.$$

The total mass of the system is just the integral of the density over the whole space:

$$m = \int_{\mathbb{R}^3} \rho \mathrm{d}^3 x.$$

This integral is finite as $\rho$ is assumed to be of compact support. Now, the total energy of the system is then given (using special relativistic arguments) by

$$\begin{aligned} E_{total} &= mc^2 + E_{grav} \\ &= c^2 \int_{\mathbb{R}^3} \rho \mathrm{d}^3 x + \frac{1}{2} \int_{\mathbb{R}^3} \rho\phi \mathrm{d}^3 x \\ &= c^2 \int_{\mathbb{R}^3} \rho \mathrm{d}^3 x + \frac{1}{8\pi G} \int_{\mathbb{R}^3} \phi\Delta\phi \mathrm{d}^3 x \\ &= c^2 \int_{\mathbb{R}^3} \rho \mathrm{d}^3 x - \frac{1}{8\pi G} \int_{\mathbb{R}^3} |\nabla\phi|^2 \mathrm{d}^3 x. \end{aligned}$$

The key observation is that the second term in the last equation is negative. As a consequence, the energy is *not bounded from below*. This is a problem, as it means one could extract an infinite amount of energy out a gravitating system. General Relativity deals with this problem by postulating the Universality of Gravity —that is, the fact that gravity can act as source of itself. To understand how this mechanism could correct things, consider a modified theory of gravity which contains a non-linear term:

$$\Delta\phi = 4\pi G\bar{\rho}, \qquad \bar{\rho} = \rho - \frac{1}{8\pi G c^2}|\nabla\phi|^2.$$

One has that $\bar{\rho}$ is an *effective density* with the non-linear term $|\nabla\phi|^2$ being the gravitational contribution to the mass density. To analyse the implications of this new theory assume

$$\phi = -\frac{m}{r} + O(1/r^2).$$

It follows that

$$4\pi m = \int_{\mathcal{S}_\infty} \nabla\phi \cdot \vec{n} \mathrm{d}S.$$

In what follows it is convenient to consider the auxiliary quantity

$$\psi = e^{\phi/2c^2}, \qquad \nabla\psi = \frac{1}{2c^2}\psi\nabla\phi.$$

Moreover,

$$\begin{aligned} \Delta\psi &= \frac{1}{4c^4}|\nabla\phi|^2 e^{\phi/2c^2} + \frac{1}{2c^2} e^{\phi/2c^2}\Delta\phi \\ &= \frac{2\pi G}{c^2}\rho\psi. \end{aligned}$$

The key observation is that $\psi$ satisfies an equation similar to the Yamabe equation. It can then be verified that

$$\begin{aligned} 4\pi m &= \int_{\mathcal{S}_\infty} \nabla\psi \cdot \vec{n} \mathrm{d}S = \frac{1}{2c^2} \int_{\mathcal{S}_\infty} \psi\nabla\phi \cdot \vec{n} \mathrm{d}S \\ &= \frac{1}{2c^2} \int_{\mathcal{S}} \nabla \cdot (\psi\nabla\phi) \mathrm{d}^3 x = \int_{\mathcal{S}} \nabla \cdot (\nabla\psi) \mathrm{d}^3 x \\ &= \int_{\mathbb{R}^3} \Delta\psi \mathrm{d}^3 x = \frac{2\pi G}{c^2} \int_{\mathbb{R}^3} \rho\psi \mathrm{d}^3 x > 0. \end{aligned}$$

A similar mechanism in General Relativity ensures the positivity of the energy. As a conclusion of these lectures we provide an overview of the proof of a particular case of the *mass positivity theorem* —[9, 10].

**Theorem 4.** *Consider a time symmetric initial data set for the vacuum Einstein field equations —i.e.* $K_{ij} = 0$. *Assume that* $\mathcal{S} \approx \mathbb{R}^3$ *with*

$$h_{\alpha\beta} - \delta_{\alpha\beta} = O(1/r),$$

*and that* $r_{ij}\lambda^i\lambda^j \geq 0$ *for* $\lambda^i \neq 0$. *Then*

$$E > 0.$$

Remarkably, the proof of this theorem makes use of *harmonic coordinates* $y^\alpha(x)$:

$$\Delta y^\alpha = D_i D^i y^\alpha = 0, \qquad y^\alpha = x^\alpha + O(1).$$

The existence of these coordinates is a further assumption which needs to be justified —this, however, will not concern us here. It is noticed that

$$|Dy|^2 \equiv h^{ij} D_i y^\alpha D_j y^\alpha, \qquad \Delta |Dy|^2 = 2(D_i D_j y^\alpha)(D^i D^j y^\alpha) + 2(D^i y^\alpha)(\Delta D_i y^\alpha).$$

Crucially, one notices that $\Delta$ and $D$ do not commute. Indeed, one has that

$$\Delta D_i y^\alpha = D^k D_k D_i y^\alpha = D^k D_i D_k y^\alpha = r_i{}^k D_k y^\alpha + D_i \Delta y^\alpha.$$

Thus,

$$\Delta |Dy|^2 = 2|DDy|^2 + 2 r_{ij} D^i y^\alpha D^j y^\alpha \geq 0$$

by assumption. It is recalled that for harmonic coordinates one has that $\Delta y^\alpha = 0$. This implies that

$$0 = h^{\beta\gamma}(\partial_\beta \partial_\gamma y^\alpha - \Gamma^\delta{}_{\beta\gamma}\partial_\delta y^\alpha) = -h^{\beta\gamma}\Gamma^\delta{}_{\beta\gamma}\partial_\delta y^\alpha.$$

Hence,

$$\Gamma^\alpha = h^{\beta\gamma}\Gamma^\alpha{}_{\beta\gamma} = 0.$$

The above expression can be reexpressed to show that

$$h^{\beta\gamma}(2\partial_\beta h_{\alpha\gamma} - \partial_\alpha h_{\beta\gamma}) = 0.$$

The above expressions are substituted in the expression of the ADM energy

$$\begin{aligned} E &= \frac{1}{16\pi G}\int_{\mathcal{S}_\infty} (\partial^\beta h_{\alpha\beta} - \partial_\alpha h_\beta{}^\beta n^\alpha) \mathrm{d}S \\ &= \frac{1}{32\pi G}\int_{\mathcal{S}_\infty} \partial^\alpha h_\beta{}^\beta n_\alpha \mathrm{d}S, \end{aligned} \tag{35}$$

where it has been used that

$$h_{\alpha\beta} = \delta_{\alpha\beta} + f_{\alpha\beta}, \qquad f_{\alpha\beta} = O(1/r)$$

so that

$$h^{\alpha\beta} = \delta^{\alpha\beta} - f^{\alpha\beta}.$$

From equation (35) one can apply the Gauss theorem to get

$$E = \frac{1}{32\pi G}\int_{\mathcal{S}_\infty} \partial^\alpha h_\beta{}^\beta n_\alpha \mathrm{d}S = \frac{1}{32\pi G}\int_{\mathcal{S}} \Delta h_\beta{}^\beta \mathrm{d}^3 y.$$

Moreover, it can be shown that in the coordinates being used one has

$$h^{\beta\beta} = |Dy|^2.$$

Thus, one concludes

$$E = \frac{1}{32\pi G}\int_{\mathcal{S}} \Delta |Dy|^2 \mathrm{d}^3 y \geq 0.$$

Which is what one wanted to show.

The positivity of mass theorem has also a rigidity part which will not be proved here.

**Theorem 5.** *For a time symmetric initial data set for the vacuum Einstein field equations with* $h_{\alpha\beta} - \delta_{\alpha\beta} = O(1/r)$ *if* $m = 0$, *then*

$$h_{\alpha\beta} = \delta_{\alpha\beta}.$$

Put in other words, if the mass vanishes and the initial data is regular, then one necessarily has initial data for the Minkowski spacetime.

## 9 Symmetries and the initial value problem

An issue which often arises in the analysis of the Cauchy problem for the Einstein field equations is that of encoding in the initial data the fact that the resulting spacetime will have a certain symmetry —i.e. a Killing vector. This naturally leads to the notion of *Killing initial data (KID)*.

To analyse the question raised in the previous paragraph, it is necessary to first consider some consequences of the Killing equation

$$\nabla_a \xi_b + \nabla_b \xi_a = 0.$$

Applying $\nabla^a$ to the above equation and commuting covariant derivatives one finds that

$$\begin{aligned} 0 &= \nabla^a \nabla_a \xi_b + \nabla^a \nabla_b \xi_a \\ &= \square \xi_b + \nabla_b \nabla^a \xi_a + R^c{}_a{}^a{}_b \xi_c \\ &= \square \xi_b - R^c{}_b \xi_c, \end{aligned}$$

where it has been used that $\nabla^a \xi_a = 0$. Accordingly, in vacuum one has that a Killing vector satisfies the wave equation

$$\square \xi_a = 0. \tag{36}$$

This equation is an integrability condition for the Killing equation. Notice however, that not every vector solution to the wave equation (36) is a Killing vector. A vector $\xi_a$ satisfying equation (36) will be called a *Killing vector candidate*.

Now, in what follows let

$$S_{ab} \equiv \nabla_a \xi_b + \nabla_b \xi_a,$$

and compute $\square S_{ab}$. Observe that commuting covariant derivatives and using that by assumption $R_{ab} = 0$ and $\nabla^e R^f{}_{bea} = 0$ one has that

$$\begin{aligned} \square \nabla_a \xi_b &= \nabla^e \nabla_e \nabla_a \xi_b \\ &= \nabla^e \nabla_a \nabla_e \xi_b + \nabla^e (R^f{}_{bea} \xi_f) \\ &= \nabla^e \nabla_a \nabla_e \xi_b + R^f{}_{bea} \nabla^e \xi_f, \\ &= \nabla_a \nabla^e \nabla_e \xi_b + R^f{}_e{}^e{}_a \nabla_f \xi_b + R^f{}_b{}^e{}_a \nabla_e \xi_f \\ &= \nabla_a \square \xi_b + R^f{}_b{}^e{}_a \nabla_e \xi_f. \end{aligned}$$

So that

$$\begin{aligned} \square S_{ab} &= R^e{}_a{}^f{}_b \nabla_f \xi_e + R^e{}_a{}^f{}_b \nabla_e \xi_f + \nabla_a \square \xi_b + \nabla_b \square \xi_a \\ &= R^e{}_a{}^f{}_b S_{ef} + \nabla_a \square \xi_b + \nabla_b \square \xi_a. \end{aligned}$$

Now, assume that one has a vector $\xi^a$ satisfying the wave equation (36). One has then that

$$\square S_{ab} - R^e{}_a{}^f{}_b S_{ef} = 0. \tag{37}$$

If initial data on an hypersurface $\mathcal{S}$ can be chosen such that

$$S_{ab} = 0, \quad \nabla_c S_{ab} = 0, \qquad \text{on } \mathcal{S} \tag{38}$$

then, as a consequence of the homogeneity of equation (37), it follows that necessarily $S_{ab} = 0$ in the development of $\mathcal{S}$ so that $\xi^a$ is, in fact, a Killing vector.

The conditions (38) are called the *Killing initial Data (KID) conditions*. They are conditions not only on $\xi^a$ but also on the initia data $(\mathcal{S}, h_{ij}, K_{ij})$. In order to see this better, one can perform a $3+1$ split of the conditions. As a first step one writes

$$\xi^a = N n^a + N^a, \qquad n_a N^a = 0,$$

where $N$ and $N^a$ denote the *lapse and shift of the Killing vector*. A computation then shows that the space-space components of the equation $\nabla_a \xi_b + \nabla_b \xi_a = 0$ imply

$$N K_{ij} + D_{(i} Y_{j)} = 0.$$

Moreover, taking a time derivative of the above equation and using the ADM evolution equations one finds that

$$N^k D_k K_{ij} + D_i N^k K_{kj} + D_j N^k K_{ik} + D_i D_j N = N(r_{ij} + KK_{ij} - 2K_{ik}K^k{}_j).$$

From the above expressions one can prove the following theorem:

**Theorem 6.** *Let $(\mathcal{S}, h_{ij}, K_{ij})$ denote an initial data set for the vacuum Einstein field equations. If there exists a pair $(N, N^i)$ such that*

$$NK_{ij} + D_{(i}Y_{j)} = 0,$$
$$N^k D_k K_{ij} + D_i N^k K_{kj} + D_j N^k K_{ik} + D_i D_j N = N(r_{ij} + KK_{ij} - 2K_{ik}K^k{}_j),$$

*then the development of the initial data has a Killing vector.*

The KID conditions are overdetermined. This is natural as not every spacetime admits a symmetry.

**Remark.** The KID conditions are closely related to the constraint equations and the ADM evolution equations —[**?**].

# 10 Bibliography

If required, a good general introduction to modern General Relativity is given in:

– R.M. Wald. General Relativity. Chicago University Press, 1984.

A number of introductory texts to various topics on mathematical General Relativity have appeared in recent years. Most notably,

– Y. Choquet-Bruhat. *General Relativity and Einstein's equations.* Oxford University Press, 2008.

– D. Christodoulou. *Mathematical problems of General Relativity. I.* Zürich lectures in Advanced Mathematics. European Mathematical Society, 2008.

– A.D. Rendall. *Partial differential equations in General Relativity.* Oxford University Presss, 2008.

– H. Ringström. *The Cauchy Problem in General Relativity.* European Mathematical Society, 2009.

# References


[1] R. Bartnik, *The mass of an asymptotically flat manifold*, Comm. Pure Appl. Math. , 661 (1986).

[2] T. W. Baumgarte & S. L. Shapiro, *Numerical Relativity: solving Einstein's equations on the computer*, Cambridge University Press, 2010.

[3] D. R. Brill & R. W. Lindquist, *Interaction energy in geometrostatics*, Phys. Rev. **131**, 471 (1963).

[4] Y. Choquet-Bruhat, *General Relativity and the Einstein equations*, Oxford University Press, 2008.

[5] H. Friedrich & A. D. Rendall, *The Cauchy problem for the Einstein equations*, Lect. Notes. Phys. **540**, 127 (2000).

[6] J. B. Griffiths & J. Podolský, *Exact space-times in Einstein's General Relativity*, Cambridge University Press, 2009.

[7] C. W. Misner, *The method of images in geometrodynamics*, Ann. Phys. **24**, 102 (1963).

[8] A. D. Rendall, *Partial differential equations in General Relativity*, Oxford University Press, 2008.

[9] R. Schoen & S. T. Yau, *On the proof of the positive mass conjecture in general relativity*, Comm. Math. Phys. **65**, 45 (1979).

[10] R. Schoen & S. T. Yau, *Proof of the positive mass theorem, II*, Comm. Math. Phys. **79**, 231 (1981).

[11] H. Stephani, D. Kramer, M. A. H. MacCallum, C. Hoenselaers, & E. Herlt, *Exact Solutions of Einstein's Field Equations*, Cambridge University Press, 2003, Second edition.

[12] R. M. Wald, *General Relativity*, The University of Chicago Press, 1984.