You may refer without proof to results from the course (theorems, examples, etc.).

Q1 This question is concerned with a set $C = \cap_{n=1}^{\infty} C_n$, where $C_1, C_2, \dots$ are constructed recursively as follows[(1)]. Start with the closed unit interval $C_1 = [0,1]$. For each $n = 1, 2, \dots$ the set C_n is a union of 2^{n-1} disjoint closed intervals, called *components*. The set C_{n+1} is obtained by removing from each component $[a,b]$ of C_n a middle open interval of size $(b-a)/(n+1)^2$, so that

$$[a,b] \cap C_{n+1} = \left[a\,,\ \frac{b+a}{2} - \frac{b-a}{2(n+1)^2}\right] \bigcup \left[\frac{b+a}{2} + \frac{b-a}{2(n+1)^2}\,,\ b\right].$$

For instance, $C_2 = [0, \frac{3}{8}] \cup [\frac{5}{8}, 1]$. Let for $x \in \mathbb{R}$

$$F(x) = \frac{\lambda(C \cap [0,x])}{\lambda(C)}, \quad F(x) = \frac{\lambda(C_n \cap [0,x])}{\lambda(C_n)}, \quad n \in \mathbb{N},$$

where λ is the Lebesgue measure.

(i) Is C a Borel set? What is the cardinality of C? Is there an open interval contained in C?

(ii) Determine $\lambda(C_n)$ for $n = 1, 2, \dots$ [Hint: find first the quotient $\lambda(C_{n+1})/\lambda(C_n)$.]

(iii) Prove that $\lambda(C) = \lim_{n\to\infty} \lambda(C_n)$ and calculate $\lambda(C)$ explicitly.

(iii) Show that F and F_n are continuous distribution functions of some probability measures μ and μ_n, respectively, and that μ_n weakly converge to μ as $n \to \infty$.

(iv) Calculate the density $f_n(x) = F_n'(x)$ for all x where the derivative exists.

(v) Find the limit $f(x) := \lim_{n\to\infty} f_n(x)$ for all x such that $F_n'(x)$ exists for every n.

Q1 solution (i) Each C_n is a finite union of closed intervals, hence Borel, and C is Borel as a countable intersection of Borel sets. Like for the standard Cantor set, points of C can be encoded by infinite binary sequences, hence the cardinality of C is continuum. Each component of C_n has length at not bigger than $2^{-(n-1)}$, and since this number approaches 0 as $n \to \infty$, the set C contains no intervals.
(ii) By the construction we have recursion

$$\lambda(C_{n+1}) = \left(1 - \frac{1}{(n+1)^2}\right) \lambda(C_n), \quad \lambda(C_1) = 1,$$

whence

$$\lambda(C_n) = \prod_{k=1}^{n-1} \left(1 - \frac{1}{(k+1)^2}\right) = \prod_{k=1}^{n-1} \frac{k(k+2)}{(k+1)^2} = \frac{1}{2}\left(1 + \frac{1}{n}\right).$$

(iii) Since $C_1 \supset C_2 \supset \cdots$, we have $\lambda(C) = \lim_{n\to\infty} \lambda(C_n)$ by the monotonicity property of measure. Using the result in (ii), sending $n \to \infty$ yields $\lambda(C) = \frac{1}{2}$.
(iii) For every measurable A, the function $x \mapsto \lambda(A \cap [0,x])$ is continuous because $\lambda(\{x\}) = 0$. Thus F_n, F are continuous distribution functions on $\mathbb{R}$, with $F_n(x) = F(x) = 0$ for $x \le 0$ and $F_n(x) = F(x) = 1$ for $x \ge 1$. By continuity of F, the weak convergence $\mu_n \Rightarrow \mu$ means convergence $F_n(x) \to F(x)$ for every $x \in \mathbb{R}$; and the latter is a consequence of $\lambda(C_n \cap [0,x]) \to \lambda(C \cap [0,x])$, which holds as in part (ii).
(iv) Measure μ_n is the uniform distribution on C_n, with density $f_n(x) = 0$ for $x \notin C_n$ and $f_n(x) =$

[(1)]Compare with the construction of the standard Cantor set.

$1/\lambda(C_n) = \frac{2n}{n+1}$ in the interior points of C_n. The derivative $F_n'(x)$ does not exist if x is an endpoint of a component of C_n.
(v) If x is an internal point of *every* C_n, then $f_n(x) \to f(x) = 1/\lambda(C) = 2$, if $x \in \mathbb{R} \setminus C$ we have $f_n(x) = 0 = f(x)$ for all sufficiently large n. If x is a boundary point of one of C_n's (endpoint of a component), $f_n'(x)$ does not exist for large enough n. [In fact, $f(x) = F'(x)$ everywhere with the exception of the latter countable set.]

Q2 Let $\xi_1, \xi_2, \dots$ be a sequence of independent, identically distributed random variables with mean $\mathbb{E}\xi_i = 0$ and variance $\mathrm{Var}(\xi_i) = \sigma^2 < \infty$. Let $S_0 = 0$, and $S_n = \xi_1 + \dots + \xi_n, \mathcal{F}_n = \sigma(\xi_1, \dots, \xi_n)$ for $n \in \mathbb{N}$.

(i) For positive function $\psi(n), n \in \mathbb{N}$, what are possible values for probability of the event $A = \{|S_n| > \psi(n) \text{ i.o.}\}$ (where i.o. means infinitely often)? Give examples of all possibilities.

(ii) Let $M_n = \sum\limits_{1 \le i < j \le n} \xi_i \xi_j$. Show that $(M_n,\ n \in \mathbb{N})$ is a martingale.

(iii) Let τ be a stopping time adapted to the filtration $(\mathcal{F}_n,\ n \in \mathbb{N})$, with $\mathbb{E}\,\tau < \infty$. For martingale from part (ii), give definition of the random variable M_τ and show that $\mathbb{E}\,M_\tau = 0$. [Hint: use Wald's identities].

(iv) Let
$$R_n = \frac{\max\limits_{0 \le i \le j \le n} |S_i - S_j|}{\sigma\sqrt{n}}.$$
Show that the random variables R_n converge in distribution as $n \to \infty$. You are not asked to find the limit distribution explicitly.

Q2 solution (i) Event A belongs to $\sigma(\xi_n, \xi_{n+1}, \dots)$ for every n, hence is a tail event, with probability 0 or 1, according to Kolmogorov's $0-1$ law. Let $A_n = \{|S_n| > \psi(n)\}$, then by Chebyshev's inequality
$$\mathbb{P}(A_n) < \frac{\mathrm{Var}(S_n)}{n^3} = \frac{\sigma^2 n}{\psi^2(n)},$$
so choosing $\psi(n) = n^{3/2}$ we have $\sum_n \mathbb{P}(A_n) < \infty$ hence $\mathbb{P}(A) = 0$ by the Borel-Cantelli lemma.

To illustrate the second possibility, let (S_n) be a simple symmetric random walk and $\psi(n) \equiv 1$; then $\mathbb{P}(A) = 1$, because the random walk is recurrent and $|S_n| > 1$ holds infinitely often.
(ii) Write $M_{n+1} = M_n + \xi_{n+1} S_n$, and observe that by measurability and independence
$$\mathbb{E}[M_n + \xi_{n+1} S_n | \mathcal{F}_n] = M_n + S_n\,\mathbb{E}[\xi_{n+1}|\mathcal{F}_n] = M_n + S_n\,\mathbb{E}[\xi_{n+1}] = M_n.$$
(iii) $M_\tau = \sum_{n=1}^\infty M_n 1(\tau = n)$ (where $1(\cdots)$ is indicator variable. Squaring yields,
$$S_n^2 = \sum_{j=1}^n \xi_j^2 + 2M_n$$
so
$$S_\tau^2 = \sum_{j=1}^\tau \xi_j^2 + 2M_\tau.$$
By the first Wald identity applied to i.i.d. ξ_j^2
$$\mathbb{E} \sum_{j=1}^\tau \xi_j^2 = \sigma^2\,\mathbb{E}\,\tau$$

and by the second $\mathbb{E}\, S_\tau^2 = \sigma^2\, \mathbb{E}\, \tau$, hence $\mathbb{E}\, M_\tau = 0$. [Under additional assumptions on τ or ξ_j's the result can be concluded straight from Doob's Optional Sampling theorem.]
(iv) For continuous function $x : [0,1] \to \mathbb{R}$ let $\rho_x := \sup_{0\le s\le t\le 1} |x(t) - x(s)|$, the functional called the range of function. Check that $|\rho_x - \rho_y| \le 2 \sup_{t\in[0,1]} |x(t) - y(t)|$, which implies that the range $x \mapsto \rho_x$ is a continuous functional on the metric space $C[0,1]$ of continuous functions. Let

$$X_n(t) = \frac{1}{\sigma\sqrt{n}} \left(\sum_{j=1}^{\lfloor nt \rfloor} \xi_j + (nt - \lfloor nt \rfloor)\xi_{\lfloor nt \rfloor + 1} \right),\ t \in [0,1]$$

be a (random) continuous function, whose graph is a broken line obtained by connecting the points

$$\left(\frac{k}{n}, \frac{S_k}{\sigma\sqrt{n}} \right), \quad k = 0, \dots, n.$$

The random variable R_n in question is the range of $X_n(\cdot)$. By Donsker's Invariance Principle $R_n \overset{d}{\to} \rho_B$, where ρ_B is the range of the Brownian motion on $[0,1]$.

Q3 Let $(B(t),\ t \ge 0)$ be a standard Brownian motion with natural filtration $(\mathcal{F}_t,\ t \ge 0)$. Consider $A(t) = |B(t)|$, the absolute value of the Brownian motion. The process $(A(t),\ t \ge 0)$ is called the *reflected Brownian motion*.

(i) Determine the probability density function $f_{A(t)}(x)$ of the random variable $A(t)$.

(ii) Determine the conditional probability density function of $A(t)$ given that $A(s) = x$, for $x, s > 0$.

(iii) Justify that $(A(t),\ t \ge 0)$ is a Markov process by showing that for $0 \le s < t$

$$\mathbb{E}[g(A(t))\,|\mathcal{F}_s] = \mathbb{E}[g(A(t))|\ A(s)].$$

for every bounded measurable function $g : \mathbb{R}_+ \to \mathbb{R}$.

(iv) Is the reflected Brownian motion a martingale, a submartingale, a supermartingale or none of these?

(v) For $x > 0$, let $\tau_x = \inf\{t \ge 0 : A(t) = x\}$. Show that $\tau_x < \infty$ a.s.. [Hint: you may use that $\{\tau_x \le t\} \supset \{A(t) \ge x\}$.]

Q3 solution (i) The function $x \to |x|$ is smooth and 2-to-1 everywhere (with the exception of 0), with derivative ± 1. Since $A(t) = |B(t)|$, the density of $A(t)$ is

$$f_{A(t)}(x) = \frac{2}{\sqrt{2\pi t}} \exp(-x^2/(2t)), \quad x > 0,$$

corresponding to the 'folded normal distribution'.
(ii) Using notation $p(t-s, x, y)$ for the transition density of the Brownian motion (for moving from $B(s) = x$ to $B(t) = y$) we have for $s < t, x \in \mathbb{R}, y > 0$ by symmetry of the centred normal distribution

$$f_{A(t)|B(s)=x}(y) = p(t-s,x,y) + p(t-s,x,-y) = p(t-s,-x,y) + p(t-s,-x,-y) = f_{A(t)|B(s)=-x}(y).$$

Hence by the total probability formula for $x \ge 0$

$$f_{A(t)|\,A(s)=x}(y) = f_{A(t)|B(s)=x}(y) = f_{A(t)|B(s)=-x}(y) = p(t-s,x,y) + p(t-s,x,-y).$$

(iii) The Brownian motion itself is a Markov process, therefore

$$\mathbb{E}[g(A(t))\,|\mathcal{F}_s] = \mathbb{E}[g(A(t))|B(s)].$$

But for $x \geq 0$ from part (ii)

$$\mathbb{E}[g(A(t))|B(s) = x] = \int_0^\infty g(y) f_{A(t)|B(s)=x}(y)\mathrm{d}y = \int_0^\infty g(y) f_{A(t)|A(s)=x}(y)\mathrm{d}y = E[g(A(t))|A(s) = x]$$

and because $x \geq 0$ is arbitrary

$$\mathbb{E}[g(A(t))\,|\mathcal{F}_s] = \mathbb{E}[g(A(t))|A(s)]$$

as wanted.
(iv) Since the function $x \mapsto |x|$ is convex, we may apply Jenssen's inequality to the conditional expectation to obtain for $s < t$

$$\mathbb{E}[A(t)|\mathcal{F}_s] = \mathbb{E}[|B(t)||\mathcal{F}_s] \geq |\,\mathbb{E}[B(t)|\mathcal{F}_s]| = |B(s)| = A(s),$$

where $\mathbb{E}[B(t)|\mathcal{F}_s] = B(s)$ holds because the Brownian motion is a martingale. Thus $(A(t), t \geq 0)$ is a submartingale.
(v) Using $B(t) \overset{d}{=} \sqrt{t}B(1) \sim \mathcal{N}(0,1)$

$$\mathbb{P}(\tau_x \leq t) \geq \mathbb{P}(A(t) \geq x) = \mathbb{P}(|B(t)| \geq x) = \mathbb{P}(|B(1)| \geq x/\sqrt{t}) = 2(1 - \Phi(x/\sqrt{t})),$$

where Φ is the cumulative distribution function of the $\mathcal{N}(0,1)$-distribution. Letting $t \to \infty$ we have $\Phi(x/\sqrt{t}) \to 1/2$, so $2(1 - \Phi(x/\sqrt{t})) \to 1$ and $\mathbb{P}(\tau_x \leq t) \to 1$. Now from

$$1 - \mathbb{P}(\tau_x < \infty) = \mathbb{P}(\tau_x = \infty) \leq \mathbb{P}(\tau_x > t) = 1 - \mathbb{P}(\tau_x \leq t)$$

we obtain

$$\mathbb{P}(\tau_x = \infty) = 0, \quad \mathbb{P}(\tau_x < \infty) = 1.$$