LTCC, Measure-Theoretic Probability: Examination 2019-20

You may refer without proof to results from the course (theorems, examples, etc.).

Q1 This question is concerned with a set $C = \cap_{n=1}^{\infty} C_n$, where $C_1, C_2, \ldots$ are constructed recursively as follows[(1)]. Start with the closed unit interval $C_1 = [0, 1]$. For each $n = 1, 2, \ldots$ the set C_n is a union of 2^{n-1} disjoint closed intervals, called *components*. The set C_{n+1} is obtained by removing from each component $[a, b]$ of C_n a middle open interval of size $(b-a)/(n+1)^2$, so that

$$[a,b] \cap C_{n+1} = \left[a\,,\ \frac{b+a}{2} - \frac{b-a}{2(n+1)^2}\right] \bigcup \left[\frac{b+a}{2} + \frac{b-a}{2(n+1)^2}\,,\ b\right].$$

For instance, $C_2 = [0, \frac{3}{8}] \cup [\frac{5}{8}, 1]$. Let for $x \in \mathbb{R}$

$$F(x) = \frac{\lambda(C \cap [0,x])}{\lambda(C)}, \quad F(x) = \frac{\lambda(C_n \cap [0,x])}{\lambda(C_n)}, \quad n \in \mathbb{N},$$

where λ is the Lebesgue measure.

(i) Is C a Borel set? What is the cardinality of C? Is there an open interval contained in C?

(ii) Determine $\lambda(C_n)$ for $n = 1, 2, \ldots$ [Hint: find first the quotient $\lambda(C_{n+1})/\lambda(C_n)$.]

(iii) Prove that $\lambda(C) = \lim_{n\to\infty} \lambda(C_n)$ and calculate $\lambda(C)$ explicitly.

(iii) Show that F and F_n are continuous distribution functions of some probability measures μ and μ_n, respectively, and that μ_n weakly converge to μ as $n \to \infty$.

(iv) Calculate the density $f_n(x) = F_n'(x)$ for all x where the derivative exists.

(v) Find the limit $f(x) := \lim_{n\to\infty} f_n(x)$ for all x such that $F_n'(x)$ exists for every n.

Q2 Let $\xi_1, \xi_2, \ldots$ be a sequence of independent, identically distributed random variables with mean $\mathbb{E}\xi_i = 0$ and variance $\mathrm{Var}(\xi_i) = \sigma^2 < \infty$. Let $S_0 = 0$, and $S_n = \xi_1 + \cdots + \xi_n, \mathcal{F}_n = \sigma(\xi_1, \ldots, \xi_n)$ for $n \in \mathbb{N}$.

(i) For positive function $\psi(n), n \in \mathbb{N}$, what are possible values for probability of the event $A = \{|S_n| > \psi(n) \text{ i.o.}\}$ (where i.o. means infinitely often)? Give examples of all possibilities.

(ii) Let $M_n = \sum_{1\le i<j\le n} \xi_i\xi_j$. Show that $(M_n,\ n \in \mathbb{N})$ is a martingale.

(iii) Let τ be a stopping time adapted to the filtration $(\mathcal{F}_n,\ n \in \mathbb{N})$, with $\mathbb{E}\,\tau < \infty$. For martingale from part (ii), give definition of the random variable M_τ and show that $\mathbb{E}\,M_\tau = 0$. [Hint: use Wald's identities.]

(iv) Let

$$R_n = \frac{\max_{0\le i\le j\le n} |S_i - S_j|}{\sigma\sqrt{n}}.$$

Show that the random variables R_n converge in distribution as $n \to \infty$. You are not asked to find the limit distribution explicitly.

(1) Compare with the construction of the standard Cantor set.

Q3 Let $(B(t),\ t \geq 0)$ be a standard Brownian motion with natural filtration $(\mathcal{F}_t,\ t \geq 0)$. Consider $A(t) = |B(t)|$, the absolute value of the Brownian motion. The process $(A(t),\ t \geq 0)$ is called the *reflected Brownian motion*.

(i) Determine the probability density function $f_{A(t)}(x)$ of the random variable $A(t)$.

(ii) Determine the conditional probability density function of $A(t)$ given that $A(s) = x$, for $x, s > 0$.

(iii) Justify that $(A(t),\ t \geq 0)$ is a Markov process by showing that for $0 \leq s < t$

$$\mathbb{E}[g(A(t))\,|\mathcal{F}_s] = \mathbb{E}[g(A(t))|\ A(s)].$$

for every bounded measurable function $g : \mathbb{R}_+ \to \mathbb{R}$.

(iv) Is the reflected Brownian motion a martingale, a submartingale, a supermartingale or none of these?

(v) For $x > 0$, let $\tau_x = \inf\{t \geq 0 : A(t) = x\}$. Show that $\tau_x < \infty$ a.s.. [Hint: you may use that $\{\tau_x \leq t\} \supset \{A(t) \geq x\}$.]