EBU5375 Signals and Systems: Signals in the time domain

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EBU5375 Signals and Systems

- Week 1: **Signals and systems in the time domain.**
- Week 2: Continuous-time signals in the frequency domain.
- Week 3: Discrete-time signals in the frequency domain.
- Week 4: Sampling theory and communication systems.
Week 1 is about...

1. The notion of signal.
2. Signals in the time domain.
3. The notion of system.
4. Systems in the time domain.
5. Linear time-invariant systems and the convolution.
This lecture is about:

1. The notion of signal.
2. Signals in the time domain: Basic signals, operations and properties.
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What is a signal? Physical point of view

- From a **physical point of view**, a signal is a **quantity that changes in time**, such as the voltage across a resistor or the sound pressure measured by a microphone.
- Signals can be **represented graphically** by plotting the value of the physical quantity against time.
The following signal corresponds to the sound produced by a cricket:

(You can listen to it by downloading the file *Cricket.wav* from QM+)
What is a signal? Mathematical point of view

- From a **mathematical point of view**, a signal is the same as a **function**, i.e. a relation between two variables.
- The **independent variable** is usually denoted by \( t \), since it represents time, and the **dependent variable** by \( x, y, \) or \( z \). Signals are denoted by \( x(t), y(t), z(t) \)…
What is a signal? Mathematical point of view

We can represent the relation between the dependent variable and the independent variable by using:

- Mathematical expressions, such as $x(t) = \cos(\pi t)$
- Look-up tables, for instance:

<table>
<thead>
<tr>
<th>t</th>
<th>x</th>
</tr>
</thead>
<tbody>
<tr>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>-1</td>
<td>3</td>
</tr>
<tr>
<td>0</td>
<td>4</td>
</tr>
<tr>
<td>1</td>
<td>5</td>
</tr>
<tr>
<td>2</td>
<td>5</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
</tr>
</tbody>
</table>
The signal \( x(t) = \cos(\pi t) \) can be represented as follows:
The independent variable: Continuous-time vs. discrete-time

There are two basic types of signals: continuous-time (CT) signals and discrete-time (DT) signals.

- In the case of CT signals, the **independent variable (time) is continuous**. We denote time by $t$ and signals by $x(t)$, $y(t)$..., where $t$ is a real value.

- In the case of discrete-time signals, the **independent variable (time) takes on a discrete set of values**. We denote time by $n$ and signals by $x[n]$, $y[n]$..., where $n = \ldots, -3, -2, -1, 0, 1, 2, 3, \ldots$ (integer value).
The independent variable: Continuous-time vs. discrete-time

CT signals and DT signals are plotted differently. In the following example, $x(t) = \cos(\pi t)$ and $y[n] = \cos \left( \frac{2\pi}{10} n \right)$:
Mathematically, the dependent variable is a number that can take on real values or complex values.

- When the dependent variable is real, signals are said to be real-valued (or real).
- When the dependent variable is complex, signals are said to be complex-valued (or complex).

Notice that a complex number can be described by two real numbers:

- Its real part and imaginary part.
- Or its magnitude and phase.

Similarly, a complex-valued signal can be described by two real-valued signals.
The dependent variable: Real and complex signals

An example of complex valued signal is \( x(t) = \cos(2\pi t) + j \sin(4\pi t) \), where \( j = \sqrt{-1} \). In the following figure we represent its real and imaginary parts, and its magnitude and phase.
Working with signals in computing environments

There exist several numerical computing environments that can be used for Signals and Systems, such as Matlab or Python.

In this module, we will use Matlab:

- Matlab is a numerical computing environment that allows vector and matrix manipulations, representation of data and implementation of algorithms.
- We will use Matlab to define, plot and manipulate signals.

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Representing signals with Matlab

Matlab uses **vectors** to represent signals. A vector is defined by placing a sequence of numbers within square braces:

```matlab
>> v = [1 2 3 4]

v =

    1     2     3     4

>> w = [0.1 0.2 0.3 0.4]

w =

    0.1000   0.2000   0.3000   0.4000
```

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Representing signals with Matlab

To represent a signal in Matlab, we use:

- One vector for representing time.
- One vector for representing the value of the signal.

Once defined, a signal can be plotted by using the command `stem` (for DT signals) and `plot` (for CT signals).

The following is a Matlab script that defines and represent the DT signal $x[n] = 3n$ in the time interval $-4 \leq n \leq 4$:

```matlab
n = [-4 -3 -2 -1 0 1 2 3 4]; % Variable n denotes time
x = 3*n; % Variable x is the signal value
stem(n,x) % plots x againsts n
xlabel('n') % adds text below the X-axis
ylabel('x[n]') % adds text beside the Y-axis
```

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Representing signals with Matlab

The result of the previous Matlab script is the following figure:
In order for us to represent a CT signal $x(t)$ in Matlab, we would need to define a vector $t$ that contains all the values of the independent variable. This is impossible.

Instead, we only use the values at a finite number of time instants, which we call **samples**. For instance, we can define and represent the signal $x(t) = 3t$ by taking small steps in time, say $\Delta t = 0.25$:

```matlab
% Variable t ... denotes time
t = [-1 -0.75 -0.5 -0.25 0 0.25 0.5 0.75 1];
x = 3*t; % Variable x is the signal value

% plots x against t
plot(t,x)
% adds text below the X-axis
xlabel('t')
% adds text beside the Y-axis
ylabel('x(t)')
```

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Representing signals with Matlab

The CT signal $x(t)$ is plotted as follows:

![Graph of a linear function](image-url)
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Unit step signal

The unit step in CT $u(t)$ is defined as

$$u(t) = \begin{cases} 1, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

Similarly, in DT the unit step $u[n]$ is defined as

$$u[n] = \begin{cases} 1, & n \geq 0 \\ 0, & n < 0 \end{cases}$$
Unit step signal
Square pulse

The square pulse in CT $p(t)$ is defined as

$$p(t) = \begin{cases} 1, & |t| \leq T_1 \\ 0, & t > T_1 \end{cases}$$

In DT, the square pulse $p[n]$ is defined as

$$p[n] = \begin{cases} 1, & |n| \leq N_1 \\ 0, & |n| > N_1 \end{cases}$$
Square pulse
Delta signals are zero everywhere except at the origin. The definition of delta signals in DT is very simple:

\[ \delta[n] = \begin{cases} 1, & n = 0 \\ 0, & \text{otherwise} \end{cases} \]

As you can see, this is the one of the simplest signals we can think of!
Delta signals

In CT, a delta signal is also zero everywhere except at $t = 0$. However, the value at the origin is undetermined.

Delta signals can be seen as the limit of narrowing a square pulse with area 1:
Delta signals

Mathematically, we define a CT delta as

\[ \delta(t) = \begin{cases} \infty, & t = 0 \\ 0, & \text{otherwise} \end{cases} \]

and add that its integration is one:

\[ \int_{-\infty}^{\infty} \delta(t) dt = \int_{-a}^{a} \delta(t) dt = 1 \]

We represent a continuous-time delta signal as an arrow at the origin:
Sinusoidal signals

Sinusoidal signals are defined in terms of a cosine function. In CT, a sinusoidal signal \( x(t) \) can be expressed as

\[
x(t) = A \cos(\omega t + \phi)
\]

where \( A \) is its **amplitude**, \( \omega \) its **angular frequency** and \( \phi \) its **phase**. CT sinusoidal signals are always **periodic**:

\[
A \cos(\omega t + \phi) = A \cos (\omega (t + T) + \phi)
\]

where \( T = \frac{2\pi}{\omega} \) and is called the **period**. In words, **sinusoidal signals look the same after** \( T \) **units of time**. The quantity \( f = \frac{1}{T} = \frac{\omega}{2\pi} \) is known as the frequency, so sinusoidal signals are also frequently expressed as:

\[
x(t) = A \cos(2\pi ft + \phi)
\]
Sinusoidal signals

As an example, identify the amplitude, period, angular frequency and phase of the signal \( x(t) = \cos(\pi t) \).
Sinusoidal signals

In DT, sinusoidal signals are expressed as follows:

\[ x[n] = A \cos(\Omega n + \phi) \]

In this case, we represent the angular frequency by the symbol \( \Omega \).

As an exercise, identify \( A \), \( \Omega \) and \( \phi \) in the following signal:
Real exponential signals

In CT, real exponential signals are of the form

\[ x(t) = Ce^{at} \]

where \( a \) is a real number. If \( a > 0 \), \( x(t) \) grows as \( t \) increases, whereas if \( a < 0 \) then \( x(t) \) decays.
Real exponential signals

In DT, real exponential signals are of the form

\[ x[n] = C e^{an} \]

where \( a \) is a real number.
Complex exponential signals

A CT complex exponential signal with purely imaginary exponent is of the form

\[ x(t) = e^{j\omega t} \]

By using Euler's relation, this complex exponential can be expressed as:

\[ e^{j\omega t} = \cos(\omega t) + j\sin(\omega t) \]

Therefore:

- The real part of \( x(t) \) is \( \Re[x(t)] = \cos(\omega t) \) and its imaginary part \( \Im[x(t)] = \sin(\omega t) \).
- The magnitude of \( x(t) \) is \( |x(t)| = 1 \) and its phase \( \angle x(t) = \omega t \).

CT exponential signals are periodic, with period \( T = \frac{2\pi}{\omega} \).
Complex exponential signals

Let $x(t) = e^{j\pi t}$. We can plot it as follows:
Complex exponential signals

DT complex exponentials have the form $e^{j\Omega n} = \cos(\Omega n) + j\sin(\Omega n)$. For instance, $x[n] = e^{j2\pi n/20}$ can be plotted as follows:
Basic DT signals in Matlab

In Matlab, a useful way of defining the time corresponding to DT signals is using

\[ n = n_s:1:n_e \]

where \( n_s \) is the start of the sequence and \( n_e \) is the end of the sequence.

For instance:

```matlab
>> n = -2:1:8
n = -2  -1   0   1   2   3   4   5   6   7   8
```
Here are several examples of DT signals in Matlab. As an exercise, define them and plot them by using the \texttt{stem} command.

\begin{verbatim}
n_s=0;
n_e=20;
n = n_s:1:n_e; % Time

x1 = exp(-0.2*n); % exponential signal
x2 = cos(2*pi*n/10); % sinusoidal signal
x3 = exp(j*2*pi*n/10); % complex exponential signal

x3r = real(x3); % real part of x3
x3i = imag(x3); % imaginary part of x3
x3a = abs(x3); % magnitude of x3
x3p = angle(x3); % phase of x3
\end{verbatim}
Basic CT signals in Matlab

We will define the independent variable of CT signals in Matlab by using

\[ t = t_s:dt:t_e \]

where \( dt \) is the time difference between two consecutive time instants, and \( t_s \) and \( t_e \) are, respectivey, the first and last time instants.

For instance:

\[
\begin{align*}
\text{>> } t &= -1:0.5:1 \\
t &= -1.0000 \quad -0.5000 \quad 0 \quad 0.5000 \quad 1.0000
\end{align*}
\]
Basic CT signals in Matlab

Here are several examples of CT signals in Matlab. As an exercise, define them and plot them by using the plot command.

t_s=0;
t_e=20;
dt=0.001;
t = t_s:dt:t_e; % Time

x1 = exp(-0.2*t); % exponential signal
x2 = cos(2*pi*t/10); % sinusoidal signal
x3 = exp(j*2*pi*t/10); % complex exponential signal

x3r = real(x3); % real part of x3
x3i = imag(x3); % imaginary part of x3
x3a = abs(x3); % magnitude of x3
x3p = angle(x3); % angle of x3
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Amplitud scaling

Given a CT signal $x(t)$, scaling consist of multiplying it by a scalar value $a$, producing the new signal $y(t) = ax(t)$.

Scaling is defined in an analogous way for DT signals.
Time shift

Given a CT signal $x(t)$, time shifting by $t_0$ units of time produces the new signal $y(t) = x(t - t_0)$ (DT shifting is defined in a similar way).

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Time reversal flips the time axis producing the signal $y(t) = x(-t)$. 
Time scaling expands or compresses the time axis. Signal $y(t) = x(at)$ is a compressed version of $x(t)$ if $|a| > 0$, and an expanded version if $|a| < 0$. 
Combining time operations

Consider the signal \( x(t) = u(-2t + 3) \). In order to obtain \( x(t) \) we will take the following steps:

- Define the time-shift \( y(t) = u(t + 3) \).
- Define the time scaling and reverse \( z(t) = y(-2t) \).

As we can see, \( z(t) = y(-2t) = u(-2t + 3) \) and therefore \( x(t) = z(t) \).
Combining time operations

In general, we can obtain the signal $y(t) = x(-at + t_0)$ by shifting $x(t)$ first and then by scaling and time reversing the result. Graphically, the signal $u(-2t + 3)$ can be obtained as follows:

You can see that $u(-2(1.5) + 3) = u(0)$.

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Sum

Adding two signals $x(t)$ and $y(t)$ means adding their values each time instant.
Similarly, we multiply signals by multiplying their values each time instant.
Basic operations with signals in Matlab

- Operating with signals in Matlab means operating with the vectors that represent them.
- Mathematically, signals extend from $-\infty$ to $\infty$. However, in Matlab we can only represent a finite number of samples.

The following are some examples of operations with DT signals in Matlab:

```matlab
n = -10:1:10; % definition of n
x = ones(size(n)); % x is a vector with the same size as n ... and all ones
y = 2*x; % y is a scaled version of x
z = x + y; % z is the sum of x and y
v = y.*z ; % v is the product of x and y, DO NOT FORGET THE DOT!
w = zeros(size(n)); % w is a vector with the same size as n ... and all zeros
w(11:end) = 1; % The samples 11 to 21 of w are set to 1
```

Plot $x$, $y$, $z$, $v$ and $w$ and make sure you understand these operations.
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Periodicity

We have already mentioned that CT sinusoidal and complex exponential signals are periodic. Now we will mathematically define periodicity. A signal $x(t)$ is periodic with period $T_0$ if

$$x(t) = x(t - T_0)$$

If $T_0$ is the smallest time delay satisfying it, we call it the fundamental period.
Periodicity

We can show that a CT complex exponential with angular frequency $\omega_0$ is periodic with period $T_0 = \frac{2\pi}{\omega_0}$ by doing the following simple manipulations:

$$x(t) = e^{j\omega_0 t} = e^{j\omega_0 t} e^{-j2\pi m} = e^{j(\omega_0 t - 2\pi m)}$$

$$= e^{j\omega_0 (t - m\frac{2\pi}{\omega_0})} = e^{j\omega_0 (t - T)}$$

$$= x(t - T)$$

where $m$ is an integer. The fundamental period is $\frac{2\pi}{\omega_0}$ and corresponds to $m = 1$.

(We can show that CT sinusoidal signals are periodic in a similar manner)
Periodicity

DT complex exponential and sinusoidal signals are, however, not always periodic! We will show this for a DT complex exponential:

\[ x[n] = e^{j\Omega_0 n} = e^{j2\pi m} = e^{j(\Omega_0 n - 2\pi m)} \]
\[ = e^{j\Omega_0 (n - m \frac{2\pi}{\Omega_0})} = e^{j\Omega_0 (n - N_0)} \]
\[ = x[n - N_0] \]

Notice that \( x[n - N_0] \) only makes sense when the quantity \( n - N_0 \) is an integer number. Therefore, a DT complex exponential signal will be periodic as long as \( N_0 \) is an integer number. If this is the case, its angular frequency will be \( m \frac{2\pi}{N_0} \).

(The same applies for DT sinusoidal signals)
Even and odd signals

A CT signal is said to be even if $x(t) = x(-t)$. Similarly, a DT signal is said to be even if $x[n] = x[-n]$.

Plot $x(t) = \cos(2\pi t)$ and check that it is even.
Even and odd signals

A CT signal is said to be odd if $x(t) = -x(-t)$. Similarly, a DT signal is said to be odd if $x[n] = -x[-n]$.

Plot $x(t) = \sin(2\pi t)$ and check that it is odd.
The area of a CT signal is defined simply as its integral:

\[ A_x = \int_{-\infty}^{\infty} x(t) \, dt \]

For DT signals, it is the sum of their values:

\[ A_x = \sum_{n=-\infty}^{\infty} x[n] \]
Average value

The average value of a CT signal is defined as:

$$\bar{x} = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} x(t) dt$$

For DT signals, it is

$$\bar{x} = \lim_{N \to \infty} \frac{1}{2N + 1} \sum_{-N}^{N} x[n]$$
The energy of a CT signal is the integral

\[ E_x = \int_{-\infty}^{\infty} |x(t)|^2 dt \]

For DT signals, it is the sum

\[ E_x = \sum_{-\infty}^{\infty} |x[n]|^2 \]
The average power of CT signals is defined as

\[ P_x = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 dt \]

For DT signals, it is defined as

\[ P_x = \lim_{N \to \infty} \frac{1}{2N + 1} \sum_{n=-N}^{N} |x[n]|^2 \]
Classification

Most signals of interest can be grouped into two broad classes:

- **Energy signals**: Signals whose energy is finite (and hence average power is zero).
- **Power signals**: Signals whose average power is not zero (and hence energy is infinite).

The square pulse is an example of energy signal and the complex exponential of power signal.
Computations with DT signals in Matlab

Calculating the area, average value, energy and mean power of DT signals involves adding samples or the square of the samples.

In Matlab, we use `sum` to add up the samples in a vector, `length` to obtain the number of samples in a vector and `x.^2` to square each sample in `x`. Here is how you calculate area, average value, energy and mean power of a DT signal `x`:

\[
\begin{align*}
\text{Ar}_x &= \text{sum}(x); \quad \% \text{Area of } x \\
\text{Av}_x &= \text{sum}(x)/\text{length}(x); \quad \% \text{Average value of } x \\
\text{E}_x &= \text{sum}(x.^2); \quad \% \text{Energy of } x, \text{ DO NOT FORGET THE DOT!} \\
\text{P}_x &= \text{sum}(x.^2)/\text{length}(x); \quad \% \text{Average value of } x, \text{ DO NOT ... } \\
&\quad \text{FORGET THE DOT!}
\end{align*}
\]

Apply these operations on the examples that we have seen before and compare the numerical result with the theoretical one.
Calculating the area, average value, energy and mean power of CT signals involves integrating. In Matlab we only represent a finite number of samples and the integral will be approximated by a sum. Given a CT signal \( x \) in Matlab and a time step between \( dt \), area, average value, energy and mean power can be calculated as follows:

\[
\begin{align*}
\text{Ar}_x &= \text{sum}(x) \times dt; \quad \% \text{ Area of } x \\
\text{Av}_x &= \frac{\text{sum}(x) \times dt}{\text{length}(x) \times dt}; \quad \% \text{ Average value of } x \\
E_x &= \text{sum}(x.^2) \times dt; \quad \% \text{ Energy of } x, \text{ DO NOT FORGET THE DOT!} \\
P_x &= \frac{\text{sum}(x.^2) \times dt}{\text{length}(x) \times dt}; \quad \% \text{ Average value of } x, \ldots \\
&\qquad \text{DO NOT FORGET THE DOT!}
\end{align*}
\]

Apply these operations to the CT signals that we have defined in our previous slides and compare the obtained values with the theoretical ones.