Let $M(\omega) = |H(e^{j\omega})|$ and $\theta(\omega) = \Phi(\omega)/2$. Also note that $M(\omega) = M(-\omega)$. $M'(\omega) = M'(\omega) = \theta'(\omega)$ and $\theta'(\omega) = \theta'(\omega)$. Therefore,

$$D = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left( |M'(\omega) + M(\omega)/\theta'(\omega)|^2 + |M'(\omega) - M(\omega)/\theta'(\omega)|^2 \right) d\omega.$$ 

Now since the integrand is positive for all $\omega$, it is sufficient to maximise the integrand to maximise $D$. Therefore,

$$\frac{d}{d\omega} \left( |M'(\omega) + M(\omega)/\theta'(\omega)|^2 + |M'(\omega) - M(\omega)/\theta'(\omega)|^2 \right) = 0.$$ 

Simplifying this, we obtain

$$2M'\theta(\omega) = 0 \Rightarrow \theta(\omega) = 0.$$ 

However, since $\theta(\omega)$ is odd, the only function that satisfies $\theta(\omega) = 0$ is $\theta(\omega) = 0$.

6.64. (a) From Table 5.1 we know that when a signal is real and even, then its Fourier transform is also real and even. Therefore, using duality, we may say that if the Fourier transform of a signal is real and even, then the signal is real and even. Therefore, $h[n] = h[-n]$. By using the time shift property, we know that if $H(e^{j\omega}) = H_0(e^{j\omega}e^{-j\pi/2})$, then

$$h[n] = h[n - M].$$

(b) We have

$$h[n + M - n] = h[M - n] = h[n].$$

Also,

$$h[M - n] = h_0[M - M - n] = h_0[-n].$$

Since $h_0[-n] = h_0[-(-n)] = h_0[-n]$, we have

$$h[M - n] = h_0[-n].$$

(c) Since $M$ is causal, $h[k] = 0$ for $k < 0$. But due to the symmetry property, $h[-k] = h_0[-(-k)] = h_0[k] = h_0[2M - k]$. Therefore, $h[k + 2M] = 0$ for $k > 0$. It follows that $h[n] = 0$ for $n > 2M$.

6.65. (a) We have

$$|H(e^{j\omega})|^2 = \frac{1}{1 + 2\cos(\omega/2)} = \frac{1}{\cos(\omega/2)} = \cos(\omega/2).$$

Figure 6.66

![Figure 6.66](image)

(d) In order for $h[n]$ to be the impulse response of an identity system, we require that $h[n] = \delta[n]$. From part (c), we know that

$$h_0[n] = h_0[2M - n] = \sum_{k=0}^{N-1} \delta[n - kN].$$

Therefore, the necessary and sufficient condition for $h[n]$ to be of $\delta[n]$ is

$$h_0[0] = \frac{1}{N}$$

and

$$h_0[kN] = 0 \text{ for } k \neq \pm 1, \pm 2, \ldots.$$
7.6. Consider the signal $u(t) = x_1(1/2)t$. The Fourier transform $W(u)$ of $u(t)$ is given by

$$W(u) = \frac{1}{\sqrt{2\pi}} X(u) \ast X(u)\big)$$

Since $X(u) = 0$ for $|u| > w_0$ and $X(u) = 0$ for $|u| > w_0$, we may conclude that $W(u) = 0$ for $|u| > w_0 + w_o$. Consequently, the Nyquist rate for $u(t)$ is $w_o = 2(w_0 + w_o)$.

Therefore, the maximum sampling period which would still allow $u(t)$ to be recovered is $T = 2\pi/w_o = \pi/(w_0 + w_o)$.

7.7. We note that $x_i(t) = h_i(t) \ast \left( \sum_{k=-\infty}^{\infty} x(kT)(t - kT) \right)$.

From Figure 7.7 in the textbook, we know that the output of the zero-order hold may be written as

$$x_o(t) = h_0(t) \ast \left( \sum_{k=-\infty}^{\infty} x(kT)(t - kT) \right).$$

where $h_0(t)$ is as shown in Figure S7.7. By taking the Fourier transform of the two above equations, we have

$$X(u) = H(u)X(u)$$

$$X(u) = H_0(u)X(u)$$

We now need to determine a frequency response $H(u)$ for a filter which produces $x_i(t)$ at its output when $x_o(t)$ is its input. Therefore, we may write

$$X(u) = H_0(u)X(u) = X_{in}(u)$$

The transfer function $H(u)$ may be obtained by convolving two rectangular pulses as shown in Figure S7.7.

Therefore,

$$h_0(t) = \frac{1}{\tau^2} H(1/\tau)(1/\tau) + (1/\tau^2)\delta(t - T/2)$$

Taking the Fourier transform of both sides of the above equation,

$$H(u) = \frac{1}{\tau^2} H(u)H(u)$$

Therefore,

$$X(u) = H(u)X(u)$$

$$X(u) = H_0(u)X(u)$$

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7.10. (a) We know that $x(t)$ is not a band-limited signal. Therefore, it cannot undergo impulse-train sampling without aliasing.

(b) From the given $X(u)$ it is clear that the signal $x(t)$ is band-limited. That is, $X(u) = 0$ for $|u| > w_o$. Therefore, it must be possible to perform impulse-train sampling on this signal without experiencing aliasing. The maximum sampling rate required would be $w_s = 2w_o$. This implies that the sampling period can at most be

$$T = 2\pi/w_s = \pi/w_o$$

(c) When $x(t)$ undergoes impulse-train sampling with $T = 2\pi/w_s$, we would obtain the signal $g(t)$ with Fourier transform

$$G(u) = \frac{1}{\tau^2} \sum_{k=-\infty}^{\infty} X(u - 2\pi k/T)$$

This is as shown in the Figure S7.10.

$$X(u)$$

$$G(u)$$

Figure S7.10

It is clear from the figure that no aliasing occurs, and that $X(u)$ can be recovered by using a filter with frequency response

$$H(u) = \frac{1}{\tau^2}, \quad 0 \leq u \leq w_o$$

Therefore, the given statement is true.

7.11. We know from Section 7.4 that

$$X_{in}(u) = \frac{1}{T} \sum_{k=-\infty}^{\infty} X(u - 2\pi k/T)$$

(a) Since $X_{in}(u)$ is just formed by shifting and summing replicas of $X(u)$, we may assume that $X_{in}(u)$ must also be real.

(b) If $X_{in}(u)$ consists of replicas of $X(u)$ which are scaled by $1/T$. Therefore, if $X(u)$ has a maximum of 1, then $X_{in}(u)$ will have a maximum of $T = 6\pi \times 10^5$.

(c) The region $-\pi/4 \leq |u| \leq \pi/4$ in the discrete-time domain corresponds to the region $-3\pi/4 \leq |u| \leq \pi/4$ in the continuous-time domain. Therefore, if $X_{in}(u)$ = $\pi/4 \leq u \leq \pi$, then $X(u)$ = 0 for $u \geq 0$. Note that $X(u)$ can be recovered if $X_{in}(u)$ is nonzero.

We already know that $X(u)$ = 0 for $u \geq 0$. Therefore, we have $X(u)$ = 0 for $u \geq 2\pi$. Since $X(u)$ = 0 for $u \geq 2\pi$, we have $X(u) = 0$ for $u \geq 2\pi$.
4. In this case, since \( \pi \) is in discrete-time frequency domain, the corresponding continuous-time frequency, \( \omega \), in the continuous-time frequency domain, which satisfies the condition given in Example 7.2, is \( \omega = \frac{\pi}{2} \). Therefore, we have \( \omega = \frac{\pi}{2} \approx 3.1416 \text{rad/sec} \). Hence, the frequency of \( x(t) \) is 3.1416 rad/sec.

For this problem, we use an approach similar to the one used in Example 7.2. We assume that

\[
x(t) = \frac{\sin(\pi t/2)}{\pi t}.
\]

The overall output is

\[
x_o(t) = x(t - \tau). \frac{\sin(\pi (t - \tau)/2)}{\pi (t - \tau)/2}.
\]

From \( x(t) \), we obtain the corresponding discrete-time signal \( x_d[n] \) to be

\[
x_d[n] = x(nT) = \frac{\sin(\pi n/2)}{\pi n/2}.
\]

Also, we obtain from \( y(t) \), the corresponding discrete-time signal \( y_d[n] \) to be

\[
y_d[n] = y(nT) = \frac{\sin(\pi n)}{\pi n}.
\]

We note that the right-hand side of the above equation is always zero when \( n \neq 0 \). When \( n = 0 \), we may evaluate the value of the ratio using L'Hopital's rule to be 1/T. Therefore, \( y_d[0] = 1/T \).

We conclude that the impulse response of the filter is

\[
h(n) = \frac{1}{n}. \quad \text{for} \quad n \neq 0,
\]

\[
h(0) = \frac{1}{T}.
\]

14. For this problem, we use an approach similar to the one used in Example 7.2. We assume that

\[
x(t) = \frac{\sin(\pi t/2)}{\pi t},
\]

The overall output is

\[
x_o(t) = x(t - \tau). \frac{\sin(\pi (t - \tau)/2)}{\pi (t - \tau)/2}.
\]

From \( x(t) \), we obtain the corresponding discrete-time signal \( x_d[n] \) to be

\[
x_d[n] = x(nT) = \frac{\sin(\pi n/2)}{\pi n/2}.
\]

Also, we obtain from \( y(t) \), the corresponding discrete-time signal \( y_d[n] \) to be

\[
y_d[n] = y(nT) = \frac{\sin(\pi n)}{\pi n}.
\]

We note that the right-hand side of the above equation is always zero when \( n \neq 0 \). When \( n = 0 \), we may evaluate the value of the ratio using L'Hopital's rule to be 1/T. Therefore, \( y_d[0] = 1/T \).

We conclude that the impulse response of the filter is

\[
h(n) = \frac{1}{n}. \quad \text{for} \quad n \neq 0,
\]

\[
h(0) = \frac{1}{T}.
\]

15. In this problem, we use the process of decimation to obtain the corresponding discrete-time signal \( y_d[n] \) to be

\[
y_d[n] = \frac{\sin(\pi (n - 1/2))}{\pi (n - 1/2)}.
\]

We note that the right-hand side of the above equation is always zero when \( n \neq 0 \). When \( n = 0 \), we may evaluate the value of the ratio using L'Hopital's rule to be 1/T. Therefore, \( y_d[0] = 1/T \).

We conclude that the impulse response of the filter is

\[
h(n) = \frac{1}{n}. \quad \text{for} \quad n \neq 0,
\]

\[
h(0) = \frac{1}{T}.
\]

16. Although the signal \( y(n) = \sin(\pi n/2)/(\pi n) \) satisfies the first two conditions, it does not satisfy the third condition. This is because the Fourier transform \( X(e^{j\omega}) \) of this signal is a rectangular pulse which is zero for \( \pi/2 < |\omega| < \pi/2 \). We also note that the signal \( |x(n)| = \sin(\pi n/2)/(\pi n)^2 \) satisfies the first two conditions. From our previous examples with this signal, we know that its Fourier transform \( X(e^{j\omega}) \) is given by the periodic convolution of \( X(e^{j\omega}) \) with itself. Therefore, \( X(e^{j\omega}) \) will be a triangular function in the range \( 0 \leq |\omega| \leq \pi \). This obviously satisfies the third condition as well. Therefore, the desired signal is \( y(n) = \sin(\pi n/2)/(\pi n)^2 \).

17. In this problem, we wish to determine the effect of decimating the impulse sequence of the given filter by a factor of 2. As explained in Section 7.5.2, the process of decimation may be broken up into two steps. In the first step, we perform impulse train sampling on \( n\) to obtain

\[
N[n] = \sum_{k=-\infty}^{\infty} \delta(2k)[n - 2k].
\]

The decimated sequence is then obtained using

\[
h_d[n] = h[n] = h[n, 2n].
\]

Using (7.37), we obtain the Fourier transform \( H_d(e^{j\omega}) \) of \( h_d[n] \) to be

\[
H_d(e^{j\omega}) = \left( 1/2 \right) H(e^{j\omega}) + \left( 1/2 \right) H(e^{j\omega - \pi}).
\]

20. (a) Suppose that \( X(e^{j\omega}) \) is as shown in Figure 7.20, then the Fourier transform \( X(e^{j\omega}) \) of the output of \( S_a \) is shown in Figure 7.21. The output \( W(e^{j\omega}) \) of the lowpass filter is as shown in Figure 7.19. The Fourier transform of the output of the decimation system \( Y(e^{j\omega}) \) is as shown in Figure 7.18. Therefore, \( Y(n) = \left( 1/2 \right) [y(n)] \).

(b) Suppose that \( X(e^{j\omega}) \) is as shown in Figure 5.20, then the Fourier transform \( X(e^{j\omega}) \) of the output of \( S_b \) is shown in Figure 5.21. The output \( W(e^{j\omega}) \) of the lowpass filter is as shown in Figure 5.19. The Fourier transform \( X(e^{j\omega}) \) of the output of \( S_a \) is shown in Figure 5.18.
of the output of the first lowpass filter are all shown in the figure below. Clearly, this system does not accomplish the filtering task.

7.21. (a) The Nyquist rate for the given signal is 2 × 5000π = 10000π. Therefore, in order to be able to recover x(t) from X(jω), the sampling period must be T_{max} = \frac{\omega}{2\pi} = 2 \times 10^{-4} sec. Since the sampling period used is T = 10^{-4} < T_{max}, x(t) can be recovered from X(jω).

(b) The Nyquist rate for the given signal is 2 × 15000π = 30000π. Therefore, in order to be able to recover x(t) from X(jω), the sampling period must be T_{max} = \frac{\omega}{2\pi} = 6 \times 10^{-4} sec. Since the sampling period used is T = 10^{-4} > T_{max}, x(t) cannot be recovered from X(jω).

(c) Here, 2πT_{max}X(jω) is not specified. Therefore, the Nyquist rate for the signal x(t) is indeterminate. This implies that we cannot guarantee that x(t) would be recoverable from X(jω).

(d) Since x(t) is real, we may conclude that X(jω) = X(jω) for |ω| > 10000π. Therefore, the answer to this part is identical to that of part (a).

(e) Since x(t) is real, X(jω) = X(jω) for |ω| > 10000π. Therefore, the answer to this part is identical to that of part (b).

(f) If X(jω) = 0 for |ω| > \omega_0, then X(jω) = 0 for |ω| > \omega_0. Therefore, in this part, X(jω) = 0 for |ω| > 7500π. The Nyquist rate for this signal is 2 × 7500π = 15000π. Therefore, in order to be able to recover x(t) from X(jω), the sampling period must be T_{max} = \frac{\omega}{2\pi} = 3 \times 10^{-4} sec. Since the sampling period used is T = 10^{-4} > T_{max}, x(t) cannot be recovered from X(jω).

7.22. Using the properties of the Fourier transform, we obtain

Y(jω) = X(jω) * X(jω).

Therefore, Y(jω) = 0 for |ω| > 10000π. This implies that the Nyquist rate for x(t) is 2 × 10000π = 20000π. Therefore, the sampling period T can at most be 2π/(20000π) = 10^{-4} sec. Therefore, we have to sample x(t) < 2πT_{max} in order to be able to recover y(t) from y(t).

7.23. (a) We may express y(t) as

y(t) = \sum_{n=-\infty}^{\infty} p(t - nT).

where p(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT).

Therefore,

P(jω) = \frac{1}{\omega} \sum_{k=-\infty}^{\infty} -j\omega k.

From this, we obtain

S(jω) = \int_{-\infty}^{\infty} |X(jω)|^2 \, d\omega = \frac{1}{\omega} \sum_{k=-\infty}^{\infty} \delta(i\omega).

Then, y(t) = \sum_{n=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \delta(t - nT - kT).

(b) Clearly, S(jω) consists of impulses spaced every 2π/T.

If Δ = T/2, then

S(jω) = \sum_{k=-\infty}^{\infty} \delta(jω - k2π/T).

Now, since X(jω) = 2πδ(jω),

W(jω) = \sum_{k=-\infty}^{\infty} \frac{1}{2\pi} \int_{-\infty}^{\infty} 2\pi \delta(j\omega - k2\pi/T) \, d\omega = 2\pi X(j\omega).

Therefore, W(jω) consists of replicas of X(jω) which are spaced 2π/T apart. In order to avoid aliasing, ω_{max} should be less than π/T. Therefore, T_{max} = \pi/ω_{max}.

If Δ = T/3, then

S(jω) = \sum_{k=-\infty}^{\infty} \delta(jω - k2\pi/3T) \, d\omega = 2\pi X(j\omega).

We note that S(jω) = 0 for n = 0, ±1, ±2, ±3, ... This is as sketched in Figure 7.24. Therefore, the replica of X(jω) in W(jω) is now spaced 2π/3T apart. In order to avoid aliasing, ω_{max} should be less than 2π/3T. Therefore, T_{max} = 3\pi/2ω_{max}.

7.25. Here, \tau_c(kT) can be written as

\tau_c(kT) = \sum_{n=-\infty}^{\infty} \tau_c(nT) \delta(x - nT).

But when n ≠ k,

\tau_c(kT) = 0,

and when n = k,

\tau_c(kT) = 1.
A system that can be used to recover $x(t)$ from $x_d(t)$ is shown in Figure 7.29.

7.29. (a) The fundamental frequency of $x(t)$ is $20\pi$ rad/sec. From Chapter 4 we know that the Fourier transform of $x(t)$ is given by

$$X(j\omega) = \frac{2\pi}{\sqrt{2\pi}} \sum_{k=-\infty}^{\infty} \delta(\omega - 2\pi k).$$

This is as sketched below. The Fourier transform $X_d(j\omega)$ of the signal $x_d(t)$ is also sketched in Figure 7.29.

Note that

$$P(j\omega) = \frac{2\pi}{\sqrt{2\pi}} \sum_{k=-\infty}^{\infty} \delta(\omega - 2\pi k)(5 \times 10^{-3})$$

and

$$X_d(j\omega) = \frac{1}{2\pi} X(j\omega) P(j\omega).$$

Therefore, $X_d(j\omega)$ is as shown in the Figure 7.29. Note that the impulses from adjacent replicas of $X_d(j\omega)$ add up to 100%. Now the Fourier transform $X_e(j\omega)$ of the envelope signal $e(t)$ is given by

$$X_e(j\omega) = X_d(j\omega) e^{-j\pi 100\pi}. $$

This is as shown in the Figure 7.29. Since the impulses in $X_e(j\omega)$ are located at multiples of $0.1\pi$, the signal $e(t)$ is periodic. The fundamental period is $2\pi/0.1\pi = 20$.

Figure 7.29

Also,

$$H(j\omega) = \frac{W(j\omega)}{X(j\omega)} = \frac{\frac{1}{2\pi} X(j\omega)}{\frac{1}{\sqrt{2\pi}} (1 - e^{-j\pi 100\pi})} = 1 - e^{-j\pi 100\pi}. $$

Therefore,

$$h(n) = \delta(n) - e^{-j\pi 100\pi}(n - 1).$$

7.31. In this problem for the sake of clarity we will use the variable $\Omega$ to denote discrete frequency.

Taking the Fourier transform of both sides of the given difference equation we obtain

$$N(\omega) = \frac{1}{2\pi} F(\omega) \frac{1 - e^{-j\pi 100\pi}}{1 - e^{-j\Omega}}. $$

Given that the sampling rate is greater than the Nyquist rate, we have

$$X(\omega) = \frac{1}{2\pi} X_e(j\Omega)/\Omega, \quad \text{for } -\pi \leq \Omega \leq \pi.$$ 

Therefore,

$$Y(\omega) = \frac{1}{1 - e^{-j\Omega}} \frac{1}{2\pi} X_e(j\Omega)/\Omega.$$ 

7.30. From Section 7.1.1 we know that

$$X_e(j\omega) = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} X(j(\omega - k\theta/\Pi))$$

and

$$X_e(j\omega) = \frac{1}{2\pi} X(j\omega) P(j\omega).$$

Therefore, $X_e(j\omega)$ is as shown in the Figure 7.29. Note that the impulses from adjacent replicas of $X_e(j\omega)$ add up to 100%. Now the Fourier transform $X_e(j\omega)$ of the envelope signal $e(t)$ is given by

$$X_e(j\omega) = X_d(j\omega) e^{-j\pi 100\pi}. $$

This is as shown in the Figure 7.29. Since the impulses in $X_e(j\omega)$ are located at multiples of $0.1\pi$, the signal $e(t)$ is periodic. The fundamental period is $2\pi/0.1\pi = 20$.

Figure 7.29

Also,

$$H(j\omega) = \frac{W(j\omega)}{X(j\omega)} = \frac{\frac{1}{2\pi} X(j\omega)}{\frac{1}{\sqrt{2\pi}} (1 - e^{-j\pi 100\pi})} = 1 - e^{-j\pi 100\pi}. $$

Therefore,

$$h(n) = \delta(n) - e^{-j\pi 100\pi}(n - 1).$$

7.31. In this problem for the sake of clarity we will use the variable $\Omega$ to denote discrete frequency.

Taking the Fourier transform of both sides of the given difference equation we obtain

$$X(\omega) = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} X(j(\omega - k\theta/\Pi)) e^{-j\pi 100\pi}.$$ 

Therefore,

$$G(\omega) = \frac{1}{2\pi} \int_{-\pi/2}^{\pi/2} p(\omega) X(e^{-j\theta/\Pi}) d\theta.$$ 

Since $X(\omega)$ is 0 for $\pi/4 \leq |\omega| \leq \pi$, $G(\omega)$ is as shown in Figure 7.32.

Figure 7.32

Clearly, in order to isolate just $X(\omega)$ we need to use an ideal lowpass filter with cutoff frequency $\pi/4$ and passband gain of 4. Therefore, in the range $|\omega| \leq \pi/4$,

$$H(\omega) = \begin{cases} 4, & |\omega| < \pi/4 \\ 0, & \pi/4 \leq |\omega| \leq \pi \end{cases}.$$ 

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For $-\pi \leq \omega \leq \pi$ from this we get

$$Y(\omega) = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} X(j(\omega - k\theta/\Pi)) e^{-j\pi 100\pi}.$$ 

for $-\pi/2 \leq \omega \leq \pi/2$. In this range, $Y(\omega)$ is $Y_1(\omega)$. Therefore,

$$H_1(\omega) = \frac{Y_1(\omega)}{X(j\omega)} = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} X(j(\omega - k\theta/\Pi)) e^{-j\pi 100\pi}. $$

7.32. Let $p(n) = \sum_{k=-\infty}^{\infty} \delta(n - k)$. Then from Chapter 5,

$$P(\omega) = e^{-j\Omega n} \sum_{k=-\infty}^{\infty} \delta(\omega - 2\pi k/4) = \sum_{k=-\infty}^{\infty} e^{-j2\pi k/4}(\delta(\omega - 2\pi k/4)).$$ 

Therefore,

$$G(\omega) = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} e^{-j2\pi k/4}(\delta(\omega - 2\pi k/4)).$$ 

Figure 7.32

Clearly, in order to isolate just $X(\omega)$ we need to use an ideal lowpass filter with cutoff frequency $\pi/4$ and passband gain of 4. Therefore, in the range $|\omega| \leq \pi/4$,

$$H(\omega) = \begin{cases} 4, & |\omega| < \pi/4 \\ 0, & \pi/4 \leq |\omega| \leq \pi \end{cases}.$$ 

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3.3. Let \( y[n] = z[n] \sum_{n=\infty}^\infty \delta[n - 4n] \). Then

\[
Y(e^{j\omega}) = \sum_{k=\infty}^\infty X(e^{j(\omega-3\pi/2)}).
\]

Note that \( u(t)(\pi/3)/(\pi/3) \) is the impulse response of an ideal lowpass filter with cutoff frequency \( \pi/2 \) and passband gain of 3. Therefore, we now require that \( y[n] \) when passed through this filter should yield \( z[n] \). Therefore, the replicas of \( X(e^{j\omega}) \) contained in \( Y(e^{j\omega}) \) should not overlap with one another. This is possible only if \( X(e^{j\omega}) = 0 \) for \( \pi/2 \leq |\omega| \leq \pi \).

3.4. In order to make \( X(e^{j\omega}) \) occupy the entire region from \(-\pi\) to \( \pi \), the signal \( x[n] \) must be downsampled by a factor of 14/3. Since this is not possible directly downsample by a noninteger factor, we first upsample the signal by a factor of 3. Therefore, after the upconversion we will need to reduce the sampling rate by 14/3 x 3 = 14. Therefore, the overall system for performing the sampling rate conversion is shown in Figure 3.35.

3.35. (a) The signals \( x[n] \) and \( x_2[n] \) are sketched in Figure 3.35.

This may be written as

\[
g(t) = 2 \text{rect}(t) + 2 \text{rect}(t - \Delta).
\]

Therefore,

\[
X(e^{j\omega}) = W \sum_{k=\infty}^\infty \left[ a + be^{-j2\pi k}\right] e^{j2\pi k}.
\]

with \( P_{1}(j\omega) \) is specified in eq. (3.37-1). Therefore,

\[
Y(e^{j\omega}) = 1/2 \text{rect}(\omega) X(e^{j\omega}).
\]

This gives

\[
y_1(t) = 2 \text{rect}(t) g(t).
\]

In the range \( 0 < \omega < W \), we may specify \( Y(e^{j\omega}) \) as

\[
Y(e^{j\omega}) = 2/\pi \left[ 1 + e^{-j\pi} \right] X(e^{j\omega}) = 2/\pi [1 + e^{-j\pi/2}].
\]

Since \( Y_1(t) = 2 \text{rect}(t) X_1(t) \), in the range \( 0 < \omega < W \), we may specify \( Y_1(t) \) as

\[
Y_1(t) = 2/\pi \left[ 1 + e^{-j\pi/2} \right] X_1(t).
\]

Since \( y_1(t) = x(t) g(t) \), in the range \( 0 < \omega < W \), we may specify \( Y_1(t) \) as

\[
Y_1(t) = 2/\pi \left[ 1 + e^{-j\pi/2} \right] X_1(t).
\]

Given that \( 0 < \Delta < \pi \), we require that \( Y_1(t) = Y_2(t) = KX(t) \) for \( 0 < \omega < W \).

Therefore,

\[
y_2(t) = 2/\pi \left[ 1 + e^{-j\pi/2} \right] X_1(t).
\]

This implies that

\[
1 + e^{-j\pi/2} = 2/\pi \left[ 1 + e^{-j\pi/2} \right] X_1(t).
\]

Solving this we obtain

\[
a = 1, \quad b = -1.
\]

When \( \Delta = \pi/2 \). More generally, we get

\[
a = \sin(\Delta) + 1 + \cos(\Delta) \quad \text{and} \quad b = -1 - \cos(\Delta) \sin(\Delta),
\]

except when \( \Delta = \pi/2 \). Finally, we also get

\[
K = 2/\pi \left[ 1 + \cos(\Delta) \right] \sin(\Delta).
\]

3.36. (a) Let us denote the sampled signal by \( y_s(t) \). We have

\[
y_s(t) = \sum_{k=\infty}^\infty y(t - kT).
\]

Since the Nyquist rate for the signal \( r(t) \) is \( 2\pi/T \), we can reconstruct the signal \( r(t) \). From Section 7.2, we know that

\[
x(t) = x_s(t) \ast h(t),
\]

where

\[
h(t) = \sin(\pi t/T) \pi/N.
\]

Therefore,

\[
x(t) = \int dt \frac{dx(t)}{dt} = x_1(t) \ast h(t).
\]

Denoting \( x_1(t) \) by \( g(t) \), we have

\[
\int dt \frac{dx(t)}{dt} = x_1(t) \ast g(t) = \sum_{k=\infty}^\infty x(t) \delta(t - kT).
\]

Therefore,

\[
p(t) = \cos(\pi t/T) - T \sin(\pi t/T) t
\]

(b) No.

3.37. We may write \( p(t) \) as

\[
p(t) = p_1(t) + p_1(t - \Delta),
\]

where

\[
p_1(t) = \sum_{k=\infty}^\infty x(t - 2kT).
\]

Therefore,

\[
p_1(t) = P_{1}(j\omega) e^{-j2\pi kW} \delta(j\omega - k\omega_0).
\]

Let us denote the product \( p_1(t)/f(t) \) by \( g(t) \). Then,

\[
g(t) = p(t)/f(t) = p_1(t)/f(t) + p_1(t - \Delta)/f(t).
\]

3.38. The Fourier transforms \( X_1(j\omega), P_{1}(j\omega) \), and \( Y(j\omega) \) are as shown in Figure 3.38.

Clearly, we cannot have \( \Delta = 0 \). Also, from the figure above it is clear that we require

\[
\frac{2\pi}{\Delta} < \frac{1}{2\pi}.
\]

This implies that

\[
\Delta < \frac{T}{4}\pi.
\]

Also from the figure, it is clear that

\[
a = \frac{2\pi}{\Delta} \frac{\Delta}{\pi}.
\]

3.39. (a) Using trigonometric identities,

\[
\cos \left( \frac{\pi}{2} \phi + \phi \right) = \cos \phi \sin \phi = \sin \left( \frac{\pi}{2} \phi \right)
\]

Therefore,

\[
g(t) = \sin \left( \frac{\pi}{2} \phi \right) \sin \phi.
\]

(b) By replacing \( \omega_0 \) with \( 2\pi/T \), and \( t \) by \( NT \) in the above equation, we get

\[
\sin \left( \frac{\pi}{2} \phi \right) \sin \phi.
\]

Clearly, the right-hand-side of the above equation is zero for \( \phi = \phi_0 \).

(c) From parts (a) and (b), we get

\[
y_s(t) = \sum_{n=\infty}^\infty \sum_{n=\infty}^\infty \sin \left( \frac{\pi}{2} \phi_0 \right) \cos \phi.
\]

From parts (a) and (b), we get

\[
y_s(t) = \sum_{n=\infty}^\infty \cos \left( \frac{\pi}{2} \phi_0 \right) \cos \phi.
\]
When this signal is passed through a lowpass filter, we are in effect performing band-limited interpolation. This results in the signal
\[ y(t) = \cos \left( \frac{\pi}{2} \right) \cos(\phi). \]

**Figure 7.40**

(a) The Fourier transform \( Y(j\omega) \) is shown in Figure 7.40.

(b) The Fourier transform \( I(j\omega) \) is
\[ I(j\omega) = \sum_{n=-\infty}^{\infty} X(j\omega - 2\pi n/T). \]

This is as shown in Figure 7.40.

(c) The Nyquist rate for \( x(t) \) is \( 2f_s \).

(d) Now,
\[ H(j\omega) = \frac{1}{1 + j\omega / \omega_s}. \]

Since \( x(t) = 2\pi(60) \sin(2\pi f_t t) \), we have \( 2f_s/T = 130\pi + 2\pi = 163\pi \). Therefore, \( H(j\omega) \) is as shown in Figure 7.40.

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(c) We require a \( T \) which avoids aliasing. Therefore, \( T < 1 / \omega_s \).

Thus, we have
\[ H(j\omega) = \frac{1}{1 + j\omega / \omega_s}. \]

For these to be consistent, we need \( A = -T \) and
\[ H(j\omega) = \frac{1}{1 + j\omega / \omega_s}. \]

For \( -\pi / 2 \leq \omega \leq \pi / 2 \).

**7.41** In this problem, to avoid confusion we use the variable \( \Omega \) to indicate discrete-time frequency.

Using Parseval's theorem and the fact that \( X(\omega) = 0 \) for \( |\omega| > \omega_s \), we get
\[ E_x = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{\sqrt{2\pi}} \int_{-\Omega_s/2}^{\Omega_s/2} |X(\Omega)|^2 d\Omega. \]

Also, using Parseval's theorem we have
\[ E_x = \sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{\sqrt{2\pi}} \int_{-\Omega_s/2}^{\Omega_s/2} |X(\Omega)|^2 d\Omega. \]

But since \( X(\omega) = X(\omega + 2\pi \Omega) \), we have
\[ E_x = \frac{1}{\sqrt{2\pi}} \int_{-\Omega_s/2}^{\Omega_s/2} |X(\Omega)|^2 d\Omega. \]

Replacing \( \Omega/T \) by \( \omega \), we get
\[ E_x = \frac{1}{2\pi} \int_{-\Omega_s/2}^{\Omega_s/2} |X(\Omega)|^2 d\Omega. \]

Also, since \( 2\pi f_s / T = 2\pi \), we may rewrite the above equation as
\[ E_x = \frac{1}{2\pi} \int_{-\Omega_s/2}^{\Omega_s/2} |X(\Omega)|^2 d\Omega = E_x. \]

**7.43** Throughout this problem, to avoid confusion we use the variable \( \Omega \) to indicate discrete-time frequency.

Taking the Fourier transform of both sides of the given differential equation, we get
\[ H(j\omega) \frac{Y(j\omega)}{X(j\omega)} = -\omega^2 + \omega + 2 \]

Therefore, \( u_k(\omega) \) obtained by passing \( r(t) \) through a lowpass filter with cutoff frequency \( 2\pi f_s \) rad/sec is
\[ u_k(\omega) = \frac{1}{2} \cos(2\pi f_s - \phi). \]

Therefore, \( \omega_k = 2\pi f_s \), \( \phi_k = -\phi \), and \( \Delta\phi = -\phi \).

(c) Here, \( 2\pi f_s = 120\pi \). Therefore, \( H(j\omega) \) is as shown in Figure 7.40.

It follows that
\[ u_k(\omega) = \frac{1}{2} \cos(2\pi f_s + \phi). \]

Therefore, if we require \( u_k(\omega) = \phi_k \) then
\[ H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = \frac{\cos(\phi)}{1 + \omega^2}. \]

Therefore, the difference equation for the filter \( h[n] \) is
\[ y[n] + a y[n-1] = x[n]. \]

(b) From Figures 7.41(a) and (b), we have
\[ h_s(\omega) = \frac{1}{\pi} \cos(\omega), \]

where \( h_s(j\omega) \) is the system response of the overall continuous-time system. Since we require that \( y_k(\omega) = \pi(\omega) \),
\[ h_s(j\omega) = \frac{1}{2\pi} \frac{Y(j\omega)}{X(j\omega)} = \frac{1}{1 + \omega^2}. \]

Comparing this with eq (7.41.1), we get \( A = 0 \).

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Taking the inverse Fourier transform of the partial fraction expansion of \( H(j\omega) \), we obtain
\[ \bar{h}(t) = \frac{1}{2\pi} \cos(\Omega(t)) - e^{\pi t} \cos(\Omega(t)). \]

Now, \( x_k(t) = \sum_{n=-\infty}^{\infty} x[n] \delta(t - nT) \).

Therefore, \( y_k(t) = X(\omega) e^{\pi T} \) for \( -\pi T \leq \omega \leq \pi T \) and 0 otherwise. From this we get
\[ \bar{y}_k(t) = H(\omega) X(\omega) e^{\pi T} \] for \( -\pi T \leq \omega \leq \pi T \) and 0 otherwise. Then, one period of \( y_k(t) \) may be specified as
\[ y_k(t) = 1/2 \bar{y}_k(t/T) \] for \( -T \leq \omega \leq T \).

Therefore, one period of \( Y(\omega) \) is
\[ Y(\omega) = X(\omega) H(\omega) e^{\pi T} \] for \( -\pi \leq \omega \leq \pi \).

Denoting the frequency response of the equivalent system by \( H(\omega) \), we have
\[ H(\omega) = H(\omega) e^{\pi T} \]

Note that \( H(\omega) \) represents the Fourier transform of the sequence \( h[n] \) obtained by lowpass filtering \( h(t) \) (with a filter of cut-off frequency \( \pi/T \)) and sampling the result every \( T \).

\[ h[n] = \sum_{k=-\infty}^{\infty} h(t) \delta(t - kT) \]

If \( \omega_n = 2\pi n/T \), then
\[ y_k(t) = \sum_{n=-\infty}^{\infty} \cos(2\pi n T) \]

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Let the range of \( T \) be \( T_{\text{min}} \leq T \leq T_{\text{max}} \). Then with \( T_{\text{max}} \), we want to obtain the smallest frequency \( \omega_0 \) and with \( T_{\text{min}} \), we want to obtain the largest frequency \( \omega_0 \). Therefore, \( T_{\text{max}} = \frac{2\pi}{N_{\omega_0}} \) and \( T_{\text{min}} = \frac{2\pi}{N_{\omega_0}} \).

(b) Let \( \phi(t) = \cos(\omega_0 t) \) and \( p(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT) \). Then

\[
\mathcal{Y}_\phi(\omega) = \frac{1}{2\pi} \left[ \mathcal{F} \{ \phi(t) \} \ast \mathcal{F} \{ p(t) \} \right].
\]

This is as shown in Figure 7.44.

![Figure 7.44](image)

(c) To avoid aliasing in \( \mathcal{Y}(\omega) \), we require that \( \omega_0 < \frac{2\pi}{T} \). Therefore, \( 4\pi N/T < 2\pi T \).

(d) The Nyquist rate for the signal \( x(t) \) is \( 6 \times 10^9 \). Therefore, the maximum value of \( T \) that can be used to sample \( x(t) \) is

\[
T_{\text{max}} = \frac{2\pi}{4 \times 10^9} = 5 \times 10^{-9}.
\]

(e) We have

\[
\lim_{n \to \infty} \mathcal{Y}(\omega) = \lim_{T \to \infty} \sum_{k=-\infty}^{\infty} \mathcal{X}(\omega) = TX(\omega^{k}).
\]

Also,

\[
\lim_{n \to \infty} x(n) = X_{\text{N}}(\omega).
\]

Therefore, eq. (7.63) requires that

\[
TX(\omega^{k}) = X_{\text{N}}(\omega).
\]

Now,

\[
X(\omega^{k}) = X_{\text{N}}(\omega^{k/T})
\]

and

\[
x(n) = \sum_{k=-\infty}^{\infty} X_{\text{N}}(\omega^{k/T}).
\]

To avoid aliasing at \( \omega_0 \) in \( \mathcal{Y}(\omega) \), we require that \( (2\pi/T) > 2\pi \times 10^{-9} \). This implies that \( T < 10^{-9} \). With this condition,

\[
X(\omega^{k}) = 1/T X_{\text{N}}(\omega^{k/T}).
\]

7.45. (a) The Nyquist rate for the signal \( x(t) \) is \( 6 \times 10^9 \). Therefore, the maximum value of \( T \) that can be used to sample \( x(t) \) is

\[
T_{\text{max}} = \frac{2\pi}{4 \times 10^9} = 5 \times 10^{-9}.
\]

(b) We have

\[
\mathcal{Y}(\omega) = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} \mathcal{X}(\omega^{k}) \mathcal{P}(\omega^{k} - \omega) = T \mathcal{X}(\omega^{k}) \mathcal{P}(\omega^{k} - \omega).
\]

Therefore,

\[
h(n) = T \mathcal{P}(n).
\]

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![Figure 7.47](image)

In order to be able to recover \( x[n] \) from \( x_p[n] \), it is clear that we need to pass \( x_p[n] \) through a lowpass filter with cutoff frequency \( \pi/3 \) and passband gain 3. Therefore,

\[
x[n] = x_p[n] \left[ \frac{3\sin(\pi n/3)}{\pi n/3} \right]
\]

\[
= \left[ \sum_{k=-\infty}^{\infty} \mathcal{X}(\omega^{k}) \mathcal{P}(\omega^{k} - \omega) \right] \left[ \frac{3\sin(\pi n/3)}{\pi n/3} \right]
\]

\[
= \sum_{k=-\infty}^{\infty} \mathcal{X}(\omega^{k}) \mathcal{P}(\omega^{k} - \omega) \sin(\pi n/3) / \pi n/3.
\]

7.48. In Figure 7.48, we plot the signal \( \mathcal{X}(\pi n/3) \).

![Figure 7.48](image)

Note that the signal \( \mathcal{X}(\pi n/3) \) contains every fourth sample of \( x[n] \). If the signal \( x[n] \) were \( \mathcal{X}(\pi n + 2\pi/3) \) (see Figure 7.48), then \( x[n] \) would be zero for all \( n \). Therefore, there would be no way of recovering \( x[n] \) from \( \mathcal{X}(\pi n/3) \). Therefore, \( \phi(n) \) should never be \( \pi/3 \) in order for the given equation to be true.

7.49. (a) Let the signals \( x_p[n] \) and \( x_q[n] \) be inputs to system A. Let the corresponding outputs be \( x_p[n] \) and \( x_q[n] \). Now, consider an input of the form \( x_p[n] = \alpha x_q[n] = \alpha x_q[n] \).

![Figure 7.49](image)

(d) \( X(\omega^{k}) \) is as sketched in Figure 7.49.

7.50. (a) We have

\[
h_0[n] = x_p[n] - x_q[n] - n \mathcal{P}(n).
\]

This is as shown in the Figure 7.50.

![Figure 7.50](image)
we have

\[ \hat{z}(\omega) = X(\omega) \hat{Y}(\omega) \]

(b) Taking the inverse Fourier transform of \( P(\omega) \), we have

\[ p(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x(\omega) e^{j\omega t} d\omega \]

From part (a), we have

\[ \hat{z}(\omega) = x(\omega) \hat{y}(\omega) \]

8.4. Consider the signal

\[ y(t) = g(t) \sin(400\pi t) \]

where

\[ g(t) = \sin(200\pi t) + \cos(400\pi t) \]

The signal \( y(t) \) is shown in Figure 85.5.

8.5. The signal \( y(t) \) is shown in Figure 85.5.

The envelope of the signal \( y(t) \) is shown in Figure 85.5. Clearly, we want to use amplitude modulation to recover the signal \( x(t) \), we need to ensure that \( A \) is greater than the height \( h \) of the highest sidelobe. The first zero-crossing of the signal \( x(t) \) occurs at time \( t_1 \) such that

\[ x(t_1) = 0 \]

Similarly, the second zero-crossing happens at time \( t_2 \) such that

\[ x(t_2) = 0 \]

The highest side-lobe occurs at time \( t_3 \), where \( x(t_3) = 0 \).

Therefore, \( A \) should be at least \( 2h \). The modulation index corresponding to the smallest permissible value of \( A \) is

\[ m = \frac{\text{Max. amplitude of } x(t)}{2h} \]

The envelope \( e(t) \) is shown in Figure 85.5.