Chapter 4 Answers

4.1 (a) Let $x(t) = e^{-|t|}u(t - 1)$. Then the Fourier transform $X(j\omega)$ of $x(t)$ is:

$$X(j\omega) = \int_{-\infty}^{\infty} e^{-|t|}u(t - 1)e^{-j\omega t}dt$$

$$= \int_{0}^{\infty} e^{-(t - 1)}e^{-j\omega t}dt + \int_{0}^{\infty} e^{-(t + 1)}e^{-j\omega t}dt$$

$$= e^{j\omega}H(2 + j\omega)$$

$|X(j\omega)|$ is as shown in Figure 4.4.1.

(b) Let $\pi(t) = e^{-2\pi(t - 1)}$. Then the Fourier transform $X(j\omega)$ of $\pi(t)$ is:

$$X(j\omega) = \int_{-\infty}^{\infty} e^{-2\pi(t - 1)}e^{-j\omega t}dt$$

$$= \int_{-\infty}^{0} e^{2\pi(1 + t)}e^{-j\omega t}dt + \int_{0}^{\infty} e^{-2\pi(1 + t)}e^{-j\omega t}dt$$

$$= e^{j\omega}H(2 + j\omega)$$

$|X(j\omega)|$ is as shown in Figure 4.4.1.

![Figure 4.4.1](image)

4.2. (a) Let $x(t) = \delta(t - 1) + \delta(t - 2)$. Then the Fourier transform $X_1(j\omega)$ of $x(t)$ is:

$$X_1(j\omega) = \int_{-\infty}^{\infty} \delta(t - 1) + \delta(t - 2)e^{-j\omega t}dt$$

$$= e^{j\omega}H(2 + j\omega)$$

$|X_1(j\omega)|$ is as shown in Figure 4.4.2.

(b) The signal $x(t) = u(-2 - t) + u(-1) - 2\delta(t - 1)$ is as shown in the figure below. Clearly,

$$\frac{d}{dt}[u(-2 - t) + u(-1) - 2\delta(t - 1)] = \delta(t - 2) - \delta(t - 1)$$

4.3. (a) The signal $x(t) = \sin(2\pi t + \pi/4)$ is periodic with a fundamental period of $T = 1$. This translates to a fundamental frequency of $\omega_0 = 2\pi$. The non-zero Fourier series coefficients of this signal may be found by writing it in the form

$$x(t) = \frac{1}{2} \left[ \sin(2\pi t + \pi) - \sin(2\pi t + 3\pi) \right]$$

$$= \frac{1}{2} \left[ \sin(2\pi t + \pi) - \sin(2\pi t + \pi/2) \right]$$

Therefore, the non-zero Fourier series coefficients of $x(t)$ are

$$a_0 = \frac{1}{2}, \quad a_1 = \frac{1}{2} \sin(2\pi t + \pi/2)$$

From Section 4.2, we know that for periodic signals, the Fourier transform consists of a train of impulses occurring at $k\omega_0$. Furthermore, the area under each impulse is $2\pi$ times the Fourier series coefficient $a_k$. Therefore, for $x(t)$, the corresponding Fourier transform $X(j\omega)$ is given by

$$X(j\omega) = 2\pi a_0 \delta(j\omega - \omega_0) + 2\pi a_1 \delta(j\omega + \omega_0)$$

$$= 2\pi \delta(j\omega - \omega_0) + \frac{\pi}{2} \cos(2\pi j\omega)$$

(b) The signal $y(t) = 1 + \exp(j\pi t/2)$ is periodic with a fundamental period of $T = 1/2$. This translates to a fundamental frequency of $\omega_0 = \pi$. The non-zero Fourier series coefficients of this signal may be found by writing it in the form

$$y(t) = 1 + \frac{\pi}{2} \sin(\pi t)$$

4.4. (a) The inverse Fourier transform is:

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega)ej\omega t d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[ 2\pi \delta(j\omega - \omega_0) + \frac{\pi}{2} \cos(2\pi j\omega) \right] ej\omega t d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[ 2\pi \delta(j\omega - \omega_0) + \frac{\pi}{2} \cos(2\pi j\omega) \right] \delta(j\omega - \omega_0) d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[ 2\pi \delta(j\omega - \omega_0) + \frac{\pi}{2} \cos(2\pi j\omega) \right] \frac{\pi}{2} \cos(2\pi j\omega) d\omega$$

$$= \frac{1}{2\pi} \left[ \frac{\pi}{2} \cos(2\pi j\omega) \right]$$

(b) The inverse Fourier transform is:

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega)ej\omega t d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[ 2\pi \delta(j\omega - \omega_0) + \frac{\pi}{2} \cos(2\pi j\omega) \right] ej\omega t d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[ 2\pi \delta(j\omega - \omega_0) + \frac{\pi}{2} \cos(2\pi j\omega) \right] \delta(j\omega - \omega_0) d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[ 2\pi \delta(j\omega - \omega_0) + \frac{\pi}{2} \cos(2\pi j\omega) \right] \frac{\pi}{2} \cos(2\pi j\omega) d\omega$$

$$= \frac{1}{2\pi} \left[ \frac{\pi}{2} \cos(2\pi j\omega) \right]$$

4.5. From the given information,$$
s(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega)ej\omega t d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[ 2\pi \delta(j\omega - \omega_0) + \frac{\pi}{2} \cos(2\pi j\omega) \right] ej\omega t d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[ 2\pi \delta(j\omega - \omega_0) + \frac{\pi}{2} \cos(2\pi j\omega) \right] \delta(j\omega - \omega_0) d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[ 2\pi \delta(j\omega - \omega_0) + \frac{\pi}{2} \cos(2\pi j\omega) \right] \frac{\pi}{2} \cos(2\pi j\omega) d\omega$$

$$= \frac{1}{2\pi} \left[ \frac{\pi}{2} \cos(2\pi j\omega) \right]$$

The signal $x(t)$ is zero where $\cos(2\pi t/3)$ is a nonzero integer multiple of $t$. This gives

$$t = \frac{k\pi}{2}, \quad k = 1, 2, 3, \ldots$$

for $k \in \mathbb{Z}$, and $k \neq 0$.  

4.6. Throughout this problem, we assume that

$$x(t) = \frac{\pi}{4} \int_{-\infty}^{\infty} X(j\omega)ej\omega t d\omega$$

(a) Using the time reversal property (Sec. 4.3.5), we have

$$x(-t) \Longleftrightarrow X(-j\omega)$$

Using the time shifting property (Sec. 4.3.3) on this, we have

$$x(t - t_1) \Longleftrightarrow e^{j\omega(t_1)}X(-j\omega)$$

$$x(t - t_2) \Longleftrightarrow e^{j\omega(t_2)}X(-j\omega)$$

Therefore,

$$x(t) = x(t - t_1 + t_2 - t_2) \Longleftrightarrow e^{j\omega(t_1 - t_2)}X(-j\omega) + e^{j\omega(t_2)}X(-j\omega)$$

(b) Using the time scaling property (Sec. 4.3.4), we have

$$x(2t) \Longleftrightarrow \frac{1}{2} X(j\omega)$$

Using the time shifting property on this, we have

$$x(2(t - t_2)) \Longleftrightarrow \frac{1}{2} X(j\omega) e^{-j\omega t_2}$$

(c) Using the differentiation in time property (Sec. 4.3.4), we have

$$\frac{dx(t)}{dt} \Longleftrightarrow \pi(j\omega X(j\omega))$$

Applying this property again, we have

$$\frac{d^2x^2(t)}{dt^2} \Longleftrightarrow \pi(j\omega X(j\omega)) e^{-j\omega t}$$

4.7. (a) Since $X(j\omega)$ is not conjugate symmetric, the corresponding signal $x(t)$ is not real. Therefore, $X(j\omega)$ is neither even nor odd, the corresponding signal $x(t)$ is neither even nor odd.

(b) The Fourier transform of a real and odd signal is purely imaginary and odd. Therefore, we can conclude that the Fourier transform of a purely imaginary and odd signal is real and odd. Since $X(j\omega)$ is real and odd, we may therefore conclude that the corresponding signal $x(t)$ is purely imaginary and odd.
(c) Consider a signal $y(t)$ whose magnitude of the Fourier transform is $|Y(j\omega)| = A(\omega)$, where the phase of the Fourier transform is $\angle Y(j\omega) = \Delta$. Since $|Y(-j\omega)| = |Y(j\omega)| = A(\omega)$ and $\angle Y(-j\omega) = -\angle Y(j\omega) = -\Delta$, we conclude that the signal $y(t)$ is real (See Table 4.1, Property 4.4.2).

Now, consider the signal $x(t)$ with Fourier transform $(\omega_1) = Y(j\omega)e^{j\omega_1 t} = \mathcal{F}(S(t))$. Using the result from the previous paragraph and the linearity property of the Fourier transform, we may conclude that $x(t)$ is imaginary. Since the Fourier transform $X(j\omega)$ is neither purely imaginary nor purely real, the signal $x(t)$ is neither even nor odd.

(d) Since $x(t)$ is both real and even, the corresponding signal $n(t)$ is real and even.

4.8. (a) The signal $n(t)$ is shown as in the Figure 4.8.

![Figure 4.8](image)

We may express this signal as

$$n(t) = \int_{-\infty}^{\infty} p(t)dt,$$

where $p(t)$ is the rectangular pulse shown in Figure 4.8. Using the integration property of the Fourier transform, we have

$$n(t) = \frac{1}{\omega} Y(j\omega) = \frac{1}{\omega} Y(j\omega) + \mathcal{F}(p(t))$$

We know from Table 4.2 that

$$Y(j\omega) = \frac{2\sin(\omega/2)}{\omega}$$

Therefore,

$$X(j\omega) = \frac{2\sin(\omega/2)}{\omega} + \pi(\omega)$$

(b) If $g(t) = \pi(t) - 1/2$, then the Fourier transform $G(j\omega)$ of $g(t)$ is given by

$$G(j\omega) = \mathcal{F}(X(j\omega)) = \int_{-\infty}^{\infty} X(j\omega)G(j\omega)$$

We have

$$G(j\omega) = \frac{2\sin(\omega/2)}{\omega} + \pi(\omega)$$

Therefore, the desired result is

$$\mathcal{F}^T[\text{Odd part of} \; n(t)] = \frac{\sin \omega}{\omega} - \frac{\cos \omega}{\omega}$$

4.10. (a) We know from Table 4.2 that

$$\int_{-\infty}^{\infty} \mathcal{F}(\sin t)$$

Therefore,

$$\int_{-\infty}^{\infty} \mathcal{F}(\sin t) = \int_{-\infty}^{\infty} \mathcal{F}(\cos t)$$

This is a triangular function $Y(j\omega)$ as shown in the Figure 4.10.

![Figure 4.10](image)

Using Table 4.1, we may write

$$\int_{-\infty}^{\infty} \mathcal{F}(\sin t) = \frac{\sin \omega}{\omega} - \frac{\cos \omega}{\omega}$$

This is as shown in the figure above. $X(j\omega)$ may be expressed mathematically as

$$X(j\omega) = \begin{cases} 1/2, & 0 \leq \omega < 0 \\ -1/2, & 0 \leq \omega < 0 \\ 0, & \text{otherwise} \end{cases}$$

(b) Using Parseval's relation,

$$\int_{-\infty}^{\infty} X(j\omega)\mathcal{F}(\sin t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(j\omega)|^2 d\omega = \frac{1}{2\pi}$$

4.9. (a) The signal $x(t)$ is plotted in Figure 4.9.

We see that the signal is very similar to the one considered in the previous problem. In fact, we may again express the signal $x(t)$ in terms of the rectangular pulse $y(t)$ shown above as follows

$$x(t) = \int_{-\infty}^{\infty} y(t)dt - u(t - \frac{1}{2})$$

Using the result obtained in part (a) of the previous problem, the Fourier transform $X(j\omega)$ of $x(t)$ is

$$X(j\omega) = \frac{2\sin(\omega/2)}{\omega} + \pi(\omega) - \mathcal{F}[u(t - \frac{1}{2})]$$

(b) The even part of $x(t)$ is given by

$$\mathcal{F}[x(t)] = \frac{\mathcal{F}[x(t)] + \mathcal{F}[-x(t)]}{2}$$

This is as shown in the Figure 4.9.

**Therefore,**

$$\mathcal{F}[x(t)] = \frac{\sin \omega}{\omega}$$

Now the real part of the answer to part (a) is

$$\Re \{ \frac{\sin \omega}{\omega} \} = \int_{-\infty}^{\infty} \mathcal{F}(x(t)x)$$

(c) The Fourier transform of the odd part of $x(t)$ is the same as $j$ times imaginary part of the answer to part (a). We have

$$2\pi \int_{-\infty}^{\infty} \mathcal{F}(x(t)x) = -\frac{\sin \omega}{\omega} + \cos \omega$$

4.11. We know that

$$x(3t) = \frac{1}{3} X(j\omega)^3, \quad x(3\omega) = \frac{1}{3} \mathcal{F}(X(j\omega)^3)$$

Therefore,

$$G(j\omega) = \mathcal{F}(x(3\omega)) = \frac{1}{3} \mathcal{F}(X(j\omega)^3)$$

Now note that

$$Y(j\omega) = \mathcal{F}(x(t) \cdot X(j\omega))$$

From this, we may write

$$Y(j\omega) = \mathcal{F}(x(3\omega) \cdot X(j\omega))$$

Using this in eq. (**), we have

$$G(j\omega) = \frac{1}{3} \mathcal{F}(X(j\omega)^3)$$

and

$$g(t) = \frac{1}{3} g(3t)$$

Therefore, $A = \frac{1}{3}$ and $B = 3$.

4.12. (a) From Example 4.2 we know that

$$e^{-\sqrt{t}} \mathcal{F} \left[ \frac{2}{1 + \omega^2} \right]$$

Using the differentiation in frequency property, we have

$$te^{-\sqrt{t}} \mathcal{F} \left[ \frac{2}{1 + \omega^2} \right] = -\frac{j\omega}{1 + \omega^2}$$

(b) The density property states that if

$$g(t) \mathcal{F} \left[ \frac{2\pi \omega}{1 + \omega^2} \right]$$

then

$$(G(j\omega)) = \frac{\mathcal{F}(2\pi \omega)}{1 + \omega^2}$$

Now, since

$$te^{-\sqrt{t}} \mathcal{F} \left[ \frac{2\pi \omega}{1 + \omega^2} \right]$$

we may use duality to write

$$\frac{j\omega}{1 + \omega^2} \mathcal{F} \left[ 2\pi \omega \right]$$

Multiplying both sides by $j$, we obtain

$$j\omega \mathcal{F} \left[ 2\pi \omega \right]$$
4.13. (a) Taking the inverse Fourier transform of \(X(\omega)\), we obtain

\[x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega\]

The signal \(x(t)\) is therefore a constant summed with two complex exponentials whose fundamental frequencies are \(\pm B\) rad/sec and \(2\) rad/sec. These two complex exponentials are not harmonically related. That is, the fundamental frequencies of these complex exponentials can never be integral multiples of a common fundamental frequency. Therefore, the signal is not periodic.

(b) Consider the signal \(y(t) = x(t) + h(t)\). From the convolution property, we know that

\[Y(\omega) = X(\omega) + H(\omega)\]

Also, from (a), we know that

\[H(\omega) = e^{-j\omega t}\sin\omega\]

The function \(H(\omega)\) is zero when \(\omega = \pm t\), where \(t\) is a nonzero integer. Therefore,

\[Y(\omega) = X(\omega) + H(\omega) = x(t) + h(t)\]

This gives

\[y(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega\]

Therefore, \(y(t)\) is a complex exponential summed with a constant. We know that a complex exponential is periodic. Adding a constant to a complex exponential does not affect its periodicity. Therefore, \(y(t)\) will be a signal with a fundamental frequency of \(2\pi/\omega\).

(c) From the results of parts (a) and (b), we see that the answer is yes.

4.14. Taking the Fourier transform of both sides of the equation

\[F^{-1}[(1 + j\omega)X(\omega)] = 2A^2 \delta(t),\]

we obtain

\[X(\omega) = \frac{A}{1 + j\omega} \delta(\omega) - \frac{1}{1 + j\omega}\]

Taking the inverse Fourier transform of the above equation

\[x(t) = Ac^{-j\omega t} \delta(t) - Ae^{-j\omega t}\]

Using Parseval's relation, we have

\[\int_{-\infty}^{\infty} |X(\omega)|^2 d\omega = 2\pi \int_{-\infty}^{\infty} |x(t)|^2 dt\]

Using the fact that \(\int_{-\infty}^{\infty} |X(\omega)|^2 d\omega = 2\pi\), we have

\[\int_{-\infty}^{\infty} |x(t)|^2 dt = 1\]

4.15. Since \(x(t)\) is real,

\[E[x(t)] = \frac{a(t) + a(-t)}{2} \int_{-\infty}^{\infty} |X(\omega)|^2 d\omega\]

We are given that

\[X(\omega) = \frac{\sin(\pi \omega/4)}{\pi \omega/4}\]

Therefore,

\[E[x(t)] = \frac{2a(t)}{\pi \omega/4} \int_{-\infty}^{\infty} \delta(\omega - \omega/4) d\omega\]

We also know that \(a(t) = 0\) for \(t < 0\). This implies that \(a(-t)\) is zero for \(t > 0\). We may conclude that

\[a(t) = 2a(t) \delta(t)\]

Therefore,

\[x(t) = 2a(t) \delta(t)\]

4.16. (a) We may write

\[x(t) = \frac{\sin(\pi \omega/4)}{\pi \omega/4} \delta(\omega - \omega/4)\]

(b) Since \(y(t)\) is an impulse train, its Fourier transform \(G(\omega)\) is also an impulse train. From Table 4.2,

\[G(\omega) = \frac{2a(t)}{\pi \omega/4} \sum_{k=0}^{\infty} \delta(\omega - 2\pi k/\omega)\]

Therefore,

\[y(t) = \sum_{k=0}^{\infty} x(\omega - k\pi/4)\]

4.18. Use the multiplication property, we know that

\[X(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega\]

If we denote \(X^T[\omega] = X(\omega)\), then

\[X(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega\]

Let us now consider the purely imaginary and odd signal \(x(\omega)\). Using linearity, we obtain the Fourier transforms of the signal to be \(jX(\omega)\). The function \(jX(\omega)\) will clearly be real and odd. Therefore, the given statement is false.

(b) An odd Fourier transform corresponds to an odd signal, while an even Fourier transform corresponds to an even signal. The convolution of an even Fourier transform with an odd Fourier may be viewed in the time domain as a multiplication of an even and odd signal. Such a multiplication will always result in an odd signal too. The Fourier transform of this odd signal will always be odd. Therefore, the given statement is true.

4.19. Using Table 4.2, we see that the rectangular pulse \(x(t)\) shown in Figure 4.18 has a Fourier transform \(X(\omega) = \sin(\omega t)/\omega\). Using the convolution property of the Fourier transform, we may write

\[x(t) = \frac{1}{2} x(t + 1) - \frac{1}{2} x(t - 1)\]

The signal \(x(t)\) is shown in Figure 4.18. Using the shifting property, we also note that

\[\frac{1}{2} x(t + 1) = \frac{1}{2} e^{j\omega t} X(\omega)\]

Mathematically \(h(t)\) may be expressed as

\[h(t) = \frac{1}{3} \delta(t) + \frac{1}{4} \delta(t - 1)\]

Since it is given that \(y(t) = e^{-j\omega t} (\sin(\omega t)\), we can compute \(Y(\omega)\) to be

\[Y(\omega) = \frac{1}{3} \omega + \frac{1}{4} \omega - \frac{1}{3} \omega (\omega + 1)\]

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Since $H(j\omega) = 1/(2 + j\omega)$, we have

$$X(j\omega) = Y(j\omega) / H(j\omega) = 1/(2 + j\omega)$$

Taking the inverse Fourier transform of $X(j\omega)$, we have

$$x(t) = e^{-\alpha t^2}$$

4.20. From the answer to Problem 3.20, we know that the frequency response of the circuit is

$$H(j\omega) = \frac{1}{\omega^2 + \alpha^2 + \frac{1}{\alpha^2} \omega + 1}$$

Breaking this up into partial fractions, we may write

$$H(j\omega) = \frac{1}{\omega^2 + \alpha^2} \left( \frac{1}{\alpha^2} \frac{1}{1 - \frac{\omega}{\alpha^2}} + \frac{1}{\alpha^2} \frac{1}{1 + \frac{\omega}{\alpha^2}} \right)$$

Using the Fourier transform pairs provided in Table 4.2, we obtain the Fourier transform of $H(j\omega)$ to be

$$h(t) = \frac{1}{\omega^2} \left( e^{-\alpha \omega^2} - \omega^2 \frac{1}{\alpha^2} e^{-\alpha \omega^2} \right)$$

4.21. (a) The given signal is

$$e^{-\alpha t} \cos(\omega_0 t) u(t) = \frac{1}{2} \left( e^{-\alpha t} \cos(\omega_0 t) + \frac{1}{\omega} e^{-\alpha t} \omega_0 \sin(\omega_0 t) \right)$$

Therefore,

$$X(j\omega) = \frac{1}{2} \left( \frac{1}{\omega^2 + \alpha^2} \right) \left( \frac{1}{\alpha^2} \frac{1}{\omega + \omega_0} + \frac{1}{\alpha^2} \frac{1}{\omega - \omega_0} \right)$$

(b) The given signal is

$$x(t) = e^{-\alpha t} \sin(2\omega_0 t) u(t) = e^{-\alpha t} \sin(2\omega_0 t)$$

Therefore,

$$x_1(t) = e^{-\alpha t} \sin(2\omega_0 t) u(t) = \frac{2\omega_0}{\alpha^2} e^{-\alpha t} \sin(2\omega_0 t)$$

Also,

$$x_2(t) = e^{-\alpha t} \sin(2\omega_0 t) u(-t) = -x_2(-t) = -x_2(t + \frac{1}{2} \omega_0)$$

Therefore,

$$X(j\omega) = \left[ \frac{2\omega_0}{\alpha^2} e^{-\alpha t} \sin(2\omega_0 t) \right] = \frac{2\omega_0}{\alpha^2} \frac{1}{\omega_0^2 + \alpha^2}$$

4.22. (a) $x(t) = e^{\alpha t}$ if $t < 0$

(b) $x(t) = \frac{1}{\alpha^2} e^{-\alpha \omega^2} \cos(t) + \frac{1}{\alpha^2} e^{-\alpha \omega^2} \sin(t)$

(c) The Fourier transform synthesis equation (4.8) may be written as

$$x(t) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega$$

From the given figure we have

$$x(t) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega$$

(d) $x(t) = \frac{1}{2} \sin(2\pi t)$

(e) Using the Fourier transform synthesis equation (4.8),

$$x(t) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega$$

23. For the given signal $x(t)$, we use the Fourier transform analysis equation (4.8) to evaluate the corresponding Fourier transform

$$X(j\omega) = x(t) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} x(t) e^{-j\omega t} d\omega$$

(i) We know that

$$x_1(t) = x(t) e^{j\omega_0 t}$$

Using the linearity and time reversal properties of the Fourier transform we have

$$X_1(j\omega) = X(j\omega) e^{-j\omega \tau}$$

4.24. (a) (i) For $X(j\omega) = 0$, the signal $x(t)$ must be real and odd. Therefore, signals in figure (a) and (c) have this property.

(ii) For $2mX(j\omega) = 0$, the signals in figure (a) and (c) have this property. (iv) For the signals in figures (a) and (c) to have this property.

(iii) For the signals in figures (a) and (c) to be real and even. Therefore, signals in figure (a) and (c) have this property. (v) For the condition to be true, $\pi \theta = 0$. Therefore, signals in figure (a), (b), (c), and (d) have this property.

(iv) For the condition to be true, the derivative of $\pi(t)$ has to be zero at $t = 0$. Therefore, signals in figure (b), (d), (e), and (f) have this property.

(v) For the signals in figure (e) to have this property. (iv) For the signals in figure (e) to have this property.

(b) For a signal to satisfy only properties (ii), (iv), and (v), it must be real and odd, and

$$x(t) = 0, \quad x'(0) = 0$$

The signal shown below is an example of that.
4.25. (a) Note that $y(t) = x(t+1)$ is a real and even signal. Therefore, $Y(j\omega)$ is also real and even. This implies that $X(j\omega) = 0$. Also, since $Y(j\omega) = e^{tx} X(j\omega)$, we know that $X(j\omega) = e^{-\omega}$. 

(b) We have 
$$X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{j\omega t} dt = 7.$$ 

(c) We have 
$$\int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} dt = 2\pi x(0) = 4e.$$ 

(d) Let $Y(j\omega) = P_{2x}(\omega)$. The corresponding signal $y(t)$ is 
$$y(t) = \begin{cases} 1 & t = 0, \\ 0 & \text{otherwise}. \end{cases}$$ 

Then the given integral is 
$$\int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} dt = 2\pi x(0) * y(t) * x(t) = 7x.$$ 

(e) We have 
$$\int_{-\infty}^{\infty} |X(j\omega)|^2 dt = 2\int_{-\infty}^{\infty} |x(t)|^2 dt = 26.$$ 

(f) The inverse Fourier transform of $X(j\omega)$ is the inverse of $X(\omega)$ which is $[x(t) + x(-t)]/2$. This is shown in the figure below.

4.26. (a) We have 
$$Y(j\omega) = X(j\omega) H(j\omega) = \frac{1}{2} \left[ \frac{1}{j\omega + \pi / 2} + \frac{1}{j\omega - \pi / 2} \right] e^{-j\pi \omega / 2}.$$ 

(b) The Fourier series coefficients $a_n$ are 
$$a_n = \frac{1}{T} \int_{0}^{T} x(t) e^{-j2\pi nt/T} dt = \frac{1}{2} \left[ \int_{0}^{\pi} x(t) e^{-j2\pi nt/T} dt - \int_{\pi}^{2\pi} x(t) e^{-j2\pi nt/T} dt \right] = \frac{\sin(n\pi)}{n\pi} e^{-j\pi n}.$$ 

Comparing the answers to parts (a) and (b), it is clear that 
$$a_n = \frac{1}{T} X[j(2\pi n/T)].$$ 

where $T = 2$. 

4.28. (a) From Table 4.2 we know that 
$$x(t) = \sum_{n=-\infty}^{\infty} x_n e^{j\omega_n t}.$$ 

From this, 
$$X(j\omega) = \sum_{n=-\infty}^{\infty} x_n X(j\omega - \omega_n) = \frac{1}{T} \left[ X(j\omega) + H(j\omega) \right].$$ 

(b) The spectra are sketched in Figure 4.28. 

4.29. (i) We have 
$$X_1(j\omega) = X(j\omega)e^{-j\pi j\omega} = X(j\omega)e^{-j\pi \omega}.$$ 

From the time shifting property we know that 
$$x_2(t) = x(t - \omega).$$ 

(ii) We have 
$$X_1(j\omega) = X(j\omega)e^{j\pi j\omega} = X(j\omega)e+j\pi \omega.$$ 

From the time shifting property we know that 
$$x_2(t) = x(t + \omega).$$ 

(iii) We have 
$$X_1(j\omega) = X(j\omega)e^{-j\pi j\omega} = X(j\omega).$$ 

From the conjugation and time reversal properties we know that 
$$x_2(t) = x(\omega - t).$$ 

Since $x(t)$ is real, $x_2(t) = x(-t)$. 

4.30. (a) We know that 
$$w(t) = \cos \frac{\pi t}{T} X(j\omega) = \delta(\omega - 1) + \delta(\omega + 1)$$ 

and 
$$y(t) = z(t) \cos \frac{\pi t}{T} X(j\omega) = \frac{1}{2} X(j\omega) + W(j\omega).$$ 

Therefore, 
$$G(j\omega) = -\frac{1}{2} X(j(\omega - 1)) + \frac{1}{2} X(j(\omega + 1)).$$
Since $G(j\omega)$ is as shown in Figure S4.30, it is clear from the above equation that $X(j\omega)$ is as shown in the Figure S4.30.

Therefore,
$$\pi(t) = \frac{2\sin t}{t^2}.$$

(b) $X_2(j\omega)$ is as shown in Figure S4.30.

4.31. (a) We have
$$x(t) = \cos \frac{2\pi t}{T}, \quad X_1(j\omega) = \pi \delta(\omega + 1) + \delta(\omega + 1).$$

(i) We have
$$h_1(t) = \text{rect}(t \leq T), \quad h_1(j\omega) = \frac{\pi}{\omega} \delta(\omega + 1) + \delta(\omega + 1).$$

Therefore,
$$Y_1(j\omega) = X_1(j\omega)H_1(j\omega) = \frac{\pi}{\omega} \delta(\omega + 1) + \delta(\omega + 1).$$

Taking the inverse Fourier transform, we obtain
$$y_1(t) = \sin(t).$$

(ii) We have
$$h_2(t) = -\delta(t + 1) + 2\delta(t) \text{rect}(t \leq T), \quad h_2(j\omega) = -\frac{\pi}{\omega} \delta(\omega + 1) - \delta(\omega + 1).$$

Therefore,
$$Y_2(j\omega) = X_1(j\omega)H_2(j\omega) = -\frac{\pi}{\omega} \delta(\omega + 1) - \delta(\omega + 1)$$

Taking the inverse Fourier transform, we obtain
$$y_2(t) = \sin(t).$$

(c) We have
$$X_3(j\omega) = \left\{ \begin{array}{ll} \sin(t) & |\omega| < 4 \\ 0 & \text{otherwise} \end{array} \right.$$\n
This implies that
$$y_3(t) = X_3(j\omega)H(j\omega) = X_3(j\omega)e^{-j\omega t}.$$

We may have obtained the same result by noting that $X_3(j\omega)$ lies entirely in the passband of $H(j\omega)$.

(d) $X_4(j\omega)$ is as shown in Figure S4.32.

Therefore,
$$Y_4(j\omega) = X_4(j\omega)H(j\omega) = X_4(j\omega)e^{-j\omega t}.$$

This implies that
$$y_4(t) = x_4(t - 1) = \frac{\sin(2\pi t - 1)}{2\pi t}.\frac{\pi}{\omega} \delta(\omega + 1) - \delta(\omega + 1).$$

We may have obtained the same result by noting that $X_4(j\omega)$ lies entirely in the passband of $H(j\omega)$.

4.33. (a) Taking the Fourier transform of both sides of the given differential equation, we obtain
$$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = -\frac{\omega}{\omega^2 + 2j\omega + 1}.$$

Using partial fraction expansion, we obtain
$$H(j\omega) = \frac{1}{\omega^2 + 2j\omega + 1} = \frac{1}{\omega^2 + 2j\omega + 1} + \frac{1}{\omega^2 + 2j\omega + 1}.$$

(b) Taking the inverse Fourier transform of both sides of the given differential equation, we obtain
$$h(t) = 2\delta(t) - \frac{2}{\pi} \sin(2\pi t).$$

4.34. (a) We have
$$h(t) = 2\delta(t) - \frac{2}{\pi} \sin(2\pi t).$$

Cross-multiplying and taking the inverse Fourier transform, we obtain
$$\frac{d^2x(t)}{dt^2} + \frac{2\pi}{\omega} \frac{dx(t)}{dt} + \frac{1}{\omega} x(t) = \delta(t) + 2\delta(t).$$

(b) We have
$$h(t) = \frac{2}{\pi} \cos(2\pi t) + \frac{1}{\pi} \sin(2\pi t).$$

Taking the inverse Fourier transform, we obtain
$$h(t) = 2\delta(t) - \frac{2}{\pi} \sin(2\pi t).$$
4.37. (a) Note that
\[ x(t) = \delta(t) + \sin(\omega t) \]
where
\[ a(t) = \begin{cases} 1, & |t| < \frac{1}{2} \\ 0, & \text{otherwise} \end{cases} \]
Also, the Fourier transform \( X(\omega) \) of \( x(t) \) is
\[ X(\omega) = \frac{\sin(\omega t)}{\omega} \]
Using the convolution property we have
\[ X(\omega) = X(\omega) * X(\omega) \]
(b) The signal \( f(t) \) is as shown in Figure 54.37.
\[ \begin{array}{c}
\text{Figure 54.37}
\end{array} \]
(c) One possible choice of \( p(t) \) is as shown in Figure 54.37.
(d) Note that
\[ \dot{f}(t) = \dot{X}(\omega(t - \frac{m}{2})) = \frac{2\pi}{m} X(\omega(t - \frac{m}{2})) \]
This may also be written as
\[ X(\omega(t - \frac{m}{2})) = \frac{2\pi}{m} \sum_{k=-\infty}^{\infty} G(jk2\pi)(\omega(t - \frac{m}{2})) \]
Clearly, this is possible only if
\[ G(jk2\pi) = X(jk2\pi) \]
\[ \text{For } n = m + 1 \text{ we may use the differentiation in frequency property to write,} \]
\[ x_{m+1}(t) = \frac{2\pi}{m} \sum_{k=-\infty}^{\infty} X(jk2\pi) \]
\[ X_{m+1}(\omega) = \frac{2\pi}{m} \sum_{k=-\infty}^{\infty} G(jk2\pi) \]
This shows that if we assume that the given statement is true for \( n = m \), then it is true for \( n = m + 1 \). Since we have shown that the given statement is true for \( n = 2 \), so the general statement is true for all \( n \).

4.41. (a) We have
\[ g(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) \sin(\omega t) \, d\omega \]
Using the frequency shift property of the Fourier transform we have
\[ \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega - \theta) \sin(\omega t) \, d\omega = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \sin(\omega t) \, d\omega \]
(b) Combining the results of parts (a) and (b),
\[ g(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} X(\omega) \sin(\omega t) \, d\omega \]
\[ y(t) = y(t) \]
4.42. \( z(t) \) is a periodic signal with Fourier series coefficients \( a_n \). The fundamental frequency of \( z(t) \) is \( \omega_f = 100 \text{ rad/sec} \). From Section 4.2 we know that the Fourier transform \( X(j\omega) \) of \( z(t) \) is
\[ X(j\omega) = \sum_{k=-\infty}^{\infty} 2\pi a_k(\omega - 100k) \]
\[ \text{Since} \]
\[ y(t) = \frac{1}{2} [X(j(\omega - \omega_0)) + X(j(\omega + \omega_0))] \]
we have
\[ Y(j\omega) = \sum_{k=-\infty}^{\infty} 2\pi a_k(\omega - 100k - \omega_0) + \sum_{k=-\infty}^{\infty} 2\pi a_k(\omega - 100k + \omega_0) \]
\[ = \sum_{k=-\infty}^{\infty} [a_k(\omega - 100k - \omega_0) + a_k(\omega - 100k + \omega_0)] \]
If \( N = 500 \), then the term in the above summation with \( k = 5 \) becomes

\[
n_{w} = \sum_{w = 0}^{N-1} a_{w} N(w) + n_{w} N(w).
\]

Since \( z(t) \) is real, \( a_{w} = a_{-w}^{*} \). Therefore, the above expression becomes \( 2 \sum_{w = 0}^{N-1} a_{w} N(w) \), which is an impulse at \( w = 0 \). Note that the inverse Fourier transform of \( 2 \sum_{w = 0}^{N-1} a_{w} N(w) \) is \( g(w) = N(w) \). Therefore, we now need to find \( \mathcal{F}(u) \) such that

\[
Y_{0}(u) = G(u) = 2 \sum_{w = 0}^{N-1} a_{w} N(w).
\]

We may easily obtain such a \( \mathcal{F}(u) \) by noting that the other terms (other than that for \( k = 5 \)) in the summation of eq. (S4.42) result in impulses at \( w = 100m \), \( m \neq 0 \). Therefore, we choose any \( \mathcal{F}(u) \) which is zero for \( w = 100m \), where \( m = \pm 1, \pm 2, \ldots \)

Similarly since

\[
\psi_{w}(t) = \left( \frac{\pi}{w} \right) \sin(\pi w t) X_{w}(t) = \frac{1}{\pi} \left( X_{w}(\pi w t) - X_{w}(\pi w t - \pi) \right)
\]

we have

\[
Y_{w}(t) = \frac{1}{\pi} \sum_{w = \pm 1}^{\infty} \left( a_{w} X_{w}(\pi w t - \pi) + a_{w} X_{w}(\pi w t) \right) + \frac{1}{\pi} \sum_{w = \pm 1}^{\infty} \left( a_{w} X_{w}(\pi w t) - a_{w} X_{w}(\pi w t - \pi) \right).
\]

Thus, we obtain

\[
Y_{w}(t) = \psi_{w}(t) = \frac{1}{\pi} \left( X_{w}(\pi w t) - X_{w}(\pi w t - \pi) \right) + \frac{1}{\pi} \sum_{\pm 1}^{\infty} a_{w} X_{w}(\pi w t).
\]

Since \( x(t) = x(t) \), we have

\[
\psi_{w}(t) = \psi_{w}(t) \Rightarrow Y_{w}(t) = X_{w}(t) N(w).
\]

Since \( Y_{w}(t) = 2 \pi \mu_{w} N(w) \), we obtain the above equation

\[
H_{w}(w) = \frac{Y_{w}(w)}{X_{w}(w)} = \left[ 2 \pi \mu_{w} N(w) \right],
\]

and

\[
\mathcal{F}_{M}(u) = \left[ 2 \pi \mu_{w} N(w) \right].
\]

4.43. Since

\[
\psi_{w}(t) = \frac{1}{\pi} \sin(\pi w t)
\]

we have

\[
Y_{w}(t) = \psi_{w}(t) = \frac{1}{\pi} \sin(\pi w t)
\]

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4.44. (a) Taking the Fourier transform of both sides of the given differential equation, we have

\[
Y_{0}(\xi) \left[ 1 + \frac{2 \pi \mu_{w} N(w)}{10 \pi \mu_{w} N(w)} \right] = X(\xi).
\]

Since \( X(\xi) = 1 + \frac{2 \pi \mu_{w} N(w)}{10 \pi \mu_{w} N(w)} \), we obtain the above equation

\[
H_{w}(w) = \frac{Y_{w}(w)}{X_{w}(w)} = \left[ 1 + \frac{2 \pi \mu_{w} N(w)}{10 \pi \mu_{w} N(w)} \right].
\]

4.45. We have

\[
Y_{w}(t) = x(t) \Rightarrow Y_{w}(t) = X_{w}(t) N(w).
\]

From Parseval's relation the total energy in \( y(t) \) is

\[
E_{y} = \int_{-\infty}^{\infty} \left| y(t) \right|^{2} dt = \int_{-\infty}^{\infty} \left| X_{w}(\xi) \right|^{2} d\xi
\]

\[
= \int_{-\infty}^{\infty} \left| X_{w}(\xi) \right|^{2} d\xi = \int_{-\infty}^{\infty} \left| X_{w}(\xi) \right|^{2} d\xi
\]

\[
= \int_{-\infty}^{\infty} \left| X_{w}(\xi) \right|^{2} d\xi + \int_{-\infty}^{\infty} \left| X_{w}(\xi) \right|^{2} d\xi
\]

For real \( x(t) \), \( \left| X_{w}(\xi) \right|^{2} = \left| X_{w}(\xi) \right|^{2} \). Therefore,

\[
E_{y} = \int_{-\infty}^{\infty} \left| X_{w}(\xi) \right|^{2} d\xi.
\]

4.46. Let \( g(t) \) be the response of \( H_{w}(w) \) to \( x(t) \). Let \( g_{w}(t) \) be the response of \( H_{w}(w) \) to \( x(t) \). Then, with reference to Figure 4.20,

\[
g_{w}(t) = g(t) = \left[ x(t) \right] \cos(w t) + \left[ x(t) \right] \sin(w t).
\]

Also,

\[
\mathcal{F}(w) = \left[ x(t) \right] \cos(w t) = \left[ x(t) \right] \sin(w t) = \left[ x(t) \right] \sin(w t)
\]

Therefore,

\[
\mathcal{F}(w) = \left[ x(t) \right] \cos(w t) + \left[ x(t) \right] \sin(w t).
\]

This is exactly what Figure 4.45 implements.

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4.47. (a) We have

\[
h_{w}(t) = \mathcal{F}(\mathcal{F}(w)) = \mathcal{F}(w).
\]

Since \( h(t) \) is causal, the new-zero portions of \( h(t) \) and \( h(-t) \) overlap only at \( t = 0 \). Therefore,

\[
h_{w}(t) = \begin{cases} 0, & t < 0 \\ h(t), & t = 0 \\ 2h(t), & t > 0 \\ \end{cases}
\]

Also, from Table 4.1 we have

\[
h_{w}(t) = \Re \mathcal{F}(\mathcal{F}(w)).
\]

4.48. Given \( \Re \mathcal{F}(\mathcal{F}(w)) \), we can obtain \( h_{w}(t) \). From \( h_{w}(t) \), we can recover \( h(t) \) (and consequently \( h(-t) \)) by using eq. (S4.57-1). Therefore, \( H(w) \) is completely specified by \( \Re \mathcal{F}(\mathcal{F}(w)) \).

(b) If

\[
\mathcal{F}(\mathcal{F}(w)) = \cos t = \frac{1}{2} \left( \delta(t) + \frac{1}{e^{t}} \right)
\]

then,

\[
h_{w}(t) = \frac{1}{2} \left( h(t) + \delta(t) \right).
\]

Therefore from eq. (S4.57-1),

\[
h(t) = \delta(t - 1).
\]

(c) We have

\[
h(t) = \frac{H(t) + H(-t)}{2}.
\]

Since \( h(t) \) is causal, the new-zero portions of \( h(t) \) and \( h(-t) \) overlap only at \( t = 0 \) and \( h_{w}(t) \) will be zero at \( t = 0 \). Therefore,

\[
h(t) = \begin{cases} 0, & t < 0 \\ \text{unknown}, & t = 0 \\ 2h(t), & t > 0 \\ \end{cases}
\]

Also, from Table 4.1 we have

\[
h_{w}(t) = \mathcal{F}(\mathcal{F}(w)).
\]

Given \( \mathcal{F}(\mathcal{F}(w)) \), we can obtain \( h_{w}(t) \). From \( h_{w}(t) \), we can recover \( h(t) \) except for \( t = 0 \) by using eq. (S4.57-1). If there are no singularities in \( h(t) \) at \( t = 0 \), then \( H(w) \) can be recovered from \( \mathcal{F}(\mathcal{F}(w)) \) even if \( h(t) \) is unknown. Therefore \( H(w) \) is completely specified by \( \mathcal{F}(\mathcal{F}(w)) \) in this case.
4.48. (a) Using the multiplication property we have
\[ h(t) = \text{rect}(t) \] 
\[ H(j\omega) = \frac{1}{2\pi} \left( H(j\omega) + \frac{1}{\omega} + \frac{1}{\omega} \right). \]
The right-hand side may be written as
\[ H(j\omega) = \frac{1}{2} \frac{1}{\omega} + \frac{1}{2\pi} \left( H(j\omega) + \frac{1}{\omega} \right). \]
That is,
\[ H(j\omega) = \frac{1}{\omega} \sum_{n=-\infty}^{\infty} H(j(\omega - 2\pi n)) \delta(t - 2\pi n). \]
Breaking up \( H(j\omega) \) into real and imaginary parts,
\[ \Re[H(j\omega)] + j\Im[H(j\omega)] = \frac{1}{\pi} \int_{-\infty}^{\infty} \Re[H(j(\omega - 2\pi n))] \delta(t - 2\pi n) \]
Comparing real and imaginary parts on both sides, we obtain
\[ H_0(j\omega) = \frac{1}{\pi} \int_{-\infty}^{\infty} \Re[H(j(\omega - 2\pi n))] \delta(t - 2\pi n) \]
From eq. (P4.45.3), we may write
\[ y(t) = x(t) * \frac{1}{\pi t} \rightarrow Y(j\omega) = X(j\omega) J_T(1/\pi t) \] (S4.45–1)
Also, from Table 4.2
\[ X(j\omega) J_T(1/\pi t) = X(j\omega) \]
Therefore,
\[ x(t) \frac{1}{\pi t} = \frac{1}{\pi t} X(j\omega) \]
Using the duality property, we have
\[ \frac{1}{\pi t} \rightarrow \frac{2}{\pi t} \]
or
\[ \frac{1}{\pi t} \rightarrow \frac{2}{\pi t} \]
Therefore, from eq. (S4.48.1), we have
\[ y(t) = x(t) H(j\omega) \]
where
\[ H(j\omega) = \frac{2}{\pi t} \quad \text{if} \quad \omega > 0, \]
\[ H(j\omega) = 0 \quad \text{if} \quad \omega = 0, \]
\[ H(j\omega) = -\frac{2}{\pi t} \quad \text{if} \quad \omega < 0. \]

(b) We may write
\[ x(t) = \int_{-\infty}^{\infty} \delta(t - \tau) \delta(\tau) \]
Therefore,
\[ x(t) = \text{rect}(t) \]
Since \( y(t) \) is real, we may write this as
\[ x(t) = X(j\omega)Y(\omega) \]
(c) Using the result of part (b) with \( y(t) \) as \( x(t) \),
\[ x(t) = X(j\omega)X(j\omega)^* \]
(d) From part (b) we have
\[ x(t) = X(j\omega)^* \]
\[ x(t) = X(j\omega)H(j\omega)X(j\omega)^* \]
\[ \Phi_{xx}(j\omega) = \Phi_{xx}(j\omega)H(j\omega) \]
Also,
\[ \Phi_{xx}(j\omega) = X(j\omega)^* \]
\[ \Phi_{xx}(j\omega) = X(j\omega)X(j\omega)^* \]
\[ = \Phi_{xx}(j\omega)H(j\omega)^2 \]
(e) From the given information, we have
\[ X(j\omega) = \frac{e^{j\omega} - 1}{j\omega} - j\frac{e^{-j\omega}}{\omega} \]
and
\[ H(j\omega) = \frac{1}{\omega} + j\omega \]
Therefore,
\[ x(t) = X(j\omega)^2 = \frac{2 - 2\cos \omega}{\omega^2} \]
\[ \Phi_{xx}(j\omega) = \Phi_{xx}(j\omega)H(j\omega) \]
and
\[ \Phi_{xx}(j\omega) = \Phi_{xx}(j\omega)H(j\omega)^2 \]

4.49. (a) (i) Since \( H(j\omega) \) is real and even, \( y(t) \) is also real and even.
\[ (i) \quad |y(t)| = \frac{1}{\pi} \int_{-\infty}^{\infty} |H(j(\omega - 2\pi n))] \delta(t - 2\pi n) \]
\[ \text{Since } H(j\omega) \text{ is real and positive,} \]
\[ |y(t)| \leq \frac{1}{\pi} \int_{-\infty}^{\infty} |H(j(\omega - 2\pi n))| \delta(t - 2\pi n) \]
\[ \text{Therefore,} \]
\[ \max |y(t)| = |y(0)| \]
(b) The bandwidth of this system is 2\( W \).
\[ (b) \quad \text{Therefore,} \]
\[ H_0(j\omega) = \frac{1}{H(j\omega)} \]
\[ \text{Therefore,} \]
\[ t_o = \frac{1}{2} + \frac{\pi}{W_0} \]
\[ \text{Therefore,} \]
\[ B_{pe} = 2W_0 - 2W \]
4.50. (a) We know from problems 1.45 and 2.67 that
\[ \Phi_{xx}(j\omega) = \Phi_{xx}(-j\omega) \]
\[ \text{Therefore,} \]
\[ \Phi_{yy}(j\omega) = \Phi_{yy}(-j\omega) \]
\[ \Phi_{yy}(j\omega) = \Phi_{yy}(j\omega) \]

4.51. (a) \( H(j\omega) = 1/G(j\omega) \).
(b) (i) If we denote the output by \( y(t) \), then we have
\[ Y(j\omega) = \frac{1}{G(j\omega)}. \]
Since \( H(j\omega) = 1/G(j\omega) \), it is impossible for us to have \( Y(j\omega) = X(j\omega)/H(j\omega) \). Therefore, we cannot find an \( x(t) \) which produces an output which looks like Figure 4.50.
(ii) This system is not invertible because \( 1/H(j\omega) \) is not defined for all \( \omega \).
(c) We have
\[ H(j\omega) = \frac{1}{G(j\omega)} \]
We now need to find a \( G(j\omega) \) such that
\[ H(j\omega)G(j\omega) = 1 \]
Thus \( G(j\omega) \) is the inverse system of \( H(j\omega) \), and is given by
\[ G(j\omega) = 1 - e^{-j\omega}G(j\omega) \]
(d) Since \( H(j\omega) = 2 + j\omega \),
\[ G(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = \frac{1}{2 + j\omega} \]
Cross-multiplying and taking the inverse Fourier transform, we obtain
\[ h(t) = \frac{1}{2} + 2\omega t + \frac{1}{2} \]
\[ H(j\omega) = \frac{-\omega^2 + 2\omega t + 2}{\omega^2 + 2\omega t + 9} \]
\[ H(j\omega) = \frac{-\omega^2 + 2\omega t + 2}{\omega^2 + 2\omega t + 9} \]
Chapter 5 Answers

5.1. (a) Let \( x[n] = (1/2)^{n-1} \cdot [n-1] \). Using the Fourier transform analysis equation (5.9), the Fourier transform \( X(e^{j\omega}) \) of this signal is

\[
X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} = \sum_{n=-\infty}^{\infty} (1/2)^{n-1} \cdot [n-1]e^{-j\omega n}
\]

(b) Let \( x[n] = (1/2)^{n} \cdot [n] \). Using the Fourier transform analysis equation (5.9), the Fourier transform \( X(e^{j\omega}) \) of this signal is

\[
X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} = \sum_{n=-\infty}^{\infty} (1/2)^{n} \cdot [n]e^{-j\omega n}
\]
5.3. We note from Section 5.2 that a periodic signal \( x[n] \) with Fourier series representation

\[
x[n] = \sum_{k=-\infty}^{\infty} a_k e^{j2\pi kn/N},
\]

has a Fourier transform

\[
X(e^{j\omega}) = \sum_{k=-\infty}^{\infty} 2\pi a_k \delta \left( \omega - \frac{2\pi k}{N} \right).
\]

(a) Consider the signal \( x[n] = \sin(\pi n) \). We note that the fundamental period of the signal \( x[n] \) is \( N = 6 \). The signal may be written as

\[
x[n] = (1/2)e^{j\pi n} + (1/2)e^{-j\pi n} = (1/2)e^{j\pi n} + (1/2)e^{-j\pi n}.
\]

From this, we obtain the non-zero Fourier coefficients \( a_k \) of \( x[n] \) in the range \(-2 \leq k \leq 3\) as

\[
a_k = (1/2)e^{j\pi k/6}, \quad a_{-k} = -(1/2)e^{-j\pi k/6}.
\]

Therefore, in the range \(-\pi \leq \omega \leq \pi \), we obtain

\[
X(e^{j\omega}) = 2\pi a_1 \delta(\omega - 2\pi/6) + 2\pi a_2 \delta(\omega - 4\pi/6) + 2\pi a_3 \delta(\omega + 2\pi/6) + 2\pi a_4 \delta(\omega + 4\pi/6).
\]

(b) Consider the signal \( x[n] = 2 + \cos(\pi n) \). We note that the fundamental period of the signal \( x[n] \) is \( N = 12 \). The signal may be written as

\[
x[n] = 2 + (1/2)e^{j\pi n} + (1/2)e^{-j\pi n} = 2 + (1/2)e^{j\pi n/12} + (1/2)e^{-j\pi n/12}.
\]

From this, we obtain the non-zero Fourier coefficients \( a_k \) of \( x[n] \) in the range \(-6 \leq k \leq 6\) as

\[
a_0 = 2, \quad a_{12} = (1/2)e^{j\pi/6}, \quad a_{-12} = (1/2)e^{-j\pi/6}.
\]

Therefore, in the range \(-\pi \leq \omega \leq \pi \), we obtain

\[
X(e^{j\omega}) = 2\pi a_0 \delta(\omega) + 2\pi a_{12} \delta(\omega - 2\pi/12) + 2\pi a_{-12} \delta(\omega + 2\pi/12).
\]

5.4. (a) Using the Fourier transform synthesis equation (5.8),

\[
x[n] = (1/2\pi) \int_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega n} \, d\omega.
\]

(b) Using the Fourier transform synthesis equation (5.8),

\[
x[n] = (1/2\pi) \int_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega n} \, d\omega.
\]

8.5. From the given information,

\[
x[n] = (1/2\pi) \int_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega n} \, d\omega.
\]

6.5. Throughout this problem, we assume that

\[
x[n] \in \mathbb{C}, \quad X(e^{j\omega}) \in \mathbb{C}.
\]

(b) We note that \( X(e^{j\omega}) \) is purely imaginary and odd. Therefore, \( x[n] \) has to be real and odd.

(c) Consider a signal \( y[n] \) whose magnitude of the Fourier transform is \( |Y(e^{j\omega})| = \delta(\omega) \), and whose phase of the Fourier transform is \( \angle Y(e^{j\omega}) = -(3/2)\omega \). Since \( |Y(e^{j\omega})| = |Y(e^{-j\omega})| \) and \( \angle Y(e^{j\omega}) = -\angle Y(e^{-j\omega}) \), we may conclude that the signal \( y[n] \) is real (see Table 5.1, Property 5.3.4). Now, consider the signal \( x[n] \) together with Fourier transform \( X(e^{j\omega}) = Y(e^{j\omega}) e^{j\omega} = -Y^*(e^{-j\omega}) \). Using the result from the previous paragraph and the linearity property of the Fourier transform, we may conclude that \( x[n] \) has to be real. Since the Fourier transform \( X(e^{j\omega}) \) is neither purely imaginary nor purely real, the signal \( x[n] \) is neither even nor odd.

5.5. Consider the signal

\[
x[n] = \begin{cases} 1, & |n| \leq 1, \\ 0, & |n| > 1. \end{cases}
\]

From Table 5.2, we know that

\[
x[n] \rightarrow X(e^{j\omega}) = \sin(3\omega/2) \over\sin(\omega/2).
\]

Using the accumulation property (Table 5.1, Property 5.3.5), we have

\[
\sum_{k=-\infty}^{\infty} x[k] e^{j\omega k} = \frac{1}{1 - e^{j\omega}} X(e^{j\omega}) + \pi \delta(\omega).
\]

Therefore, in the range \(-\pi < \omega \leq \pi\),

\[
\sum_{k=-1}^{1} x[k] e^{j\omega k} = \frac{1}{1 - e^{j\omega}} X(e^{j\omega}) + 3\pi \delta(\omega).
\]

Also, in the range \(-\pi < \omega \leq \pi\),

\[
\sum_{k=-\infty}^{\infty} x[k] e^{j\omega k} = \pi e^{j\omega} \delta(\omega).
\]

Therefore, in the range \(-\pi < \omega \leq \pi\),

\[
x[n] = 1 + \sum_{k=-1}^{1} x[k] e^{j\omega k} = \begin{cases} 1, & |n| \leq 1, \\ 0, & |n| > 1. \end{cases}
\]

The signal \( x[n] \) has the desired Fourier transform. We may express \( x[n] \) mathematically as

\[
x[n] = 1 + \sum_{k=-1}^{1} x[k] e^{j\omega k} = \begin{cases} 1, & |n| \leq 1, \\ 0, & |n| > 1. \end{cases}
\]
From Property 5.3.4 in Table 5.1, we know that for a real signal $x[n]$, 
\[ G_Y(e^{j\omega}) = \frac{1}{2\pi} G_X(e^{j\omega}) \]
From the given information, 
\[ X(\alpha e^{j\omega}) = j\sin\beta - j\sin\theta \]
\[ = (1/2\pi)e^{j\omega} - e^{j\alpha \omega} + e^{-j\alpha \omega} \]
Therefore, 
\[ G_Y(e^{j\omega}) = \frac{1}{2\pi} \left( j\sin\beta - j\sin\theta \right) = \frac{1}{2\pi} \left( \frac{1}{2\pi}e^{j\omega} - e^{j\alpha \omega} + e^{-j\alpha \omega} \right) \]
We also know that 
\[ 0 \leq \beta \leq 2\pi \]
and that $x[n] = 0$ for $n > 0$. Therefore, 
\[ x[n] = -D(\alpha) + \delta(n + 1) - \delta(n + 2) + \delta(n - 2) \]
Now we only have to find $x[0]$. Using Parseval's relation, we have 
\[ \frac{1}{2\pi} \int_{-\pi}^{\pi} |X(e^{j\omega})|^2 d\omega = \sum_{n=-\infty}^{\infty} |x[n]|^2 \]
From the given information, we can write 
\[ y^2 = \sum_{n=-\infty}^{\infty} |x[n]|^2 = \sum_{n=-\infty}^{\infty} |a[n]|^2 \]
This gives $y = \pm 1$. But since we are given that $x[0] > 0$, we conclude that $x[0] = 1$. Therefore, 
\[ x[n] = \delta(n) + \delta(n + 1) - \delta(n + 2) + \delta(n - 2) \]
5.10. From Table 5.2, we know that 
\[ \left( 1/2 \right)^n w[n] \left( \frac{1}{1 - e^{-j\omega}} \right) \]
Using Property 5.3.8 in Table 5.1, 
\[ x[n] = \left( \frac{1}{2} \right)^n w[n] \left( \frac{1}{1 - e^{-j\omega}} \right) X(e^{j\omega}) = \frac{1}{1 - e^{-j\omega}} = \frac{1}{1 - e^{j\omega}} \]
Therefore, 
\[ \sum_{n=-\infty}^{\infty} \left( \frac{1}{2} \right)^n x[n] = \sum_{n=-\infty}^{\infty} \frac{1}{2^n} x[n] = X(e^{j\omega}) = 2 \]

The plot of $\text{FT} \left\{ \frac{\sin(n\omega)}{n\omega} \right\}$ is shown in Figure 5.12. It is clear that if $Y(e^{j\omega}) = X(e^{j\omega})$, then $(\pi/2) \leq \omega \leq \pi$.

1.3. When two LTI systems are connected in parallel, the impulse response of the overall system is the sum of the impulse responses of the individual systems. Therefore, 
\[ h[n] = h_1[n] + h_2[n] \]
Using the linearity property (Table 5.1, Property 5.3.2), 
\[ X(e^{j\omega}) = H_1(e^{j\omega}) + H_2(e^{j\omega}) \]
Given that $h_1[n] = (1/2)^n w[n]$, we obtain 
\[ H_1(e^{j\omega}) = \frac{1}{1 - e^{-j\omega}} \]

Therefore, 
\[ H_2(e^{j\omega}) = \frac{1}{1 - e^{-j\omega}} \]
Taking the inverse Fourier transform, 
\[ h_2[n] = -2 \left( \frac{1}{2} \right)^n w[n] \]

1.4. From the given information, we have the Fourier transform $G(\alpha)$ of $g[n]$ to be 
\[ G(\alpha) = g[0] + g[1]e^{-j\alpha} + g[2]e^{-2j\alpha} \]
Also, when the input to the system is $x[n] = (1/4)^n w[n]$, the output is $g[n]$. Therefore, 
\[ H(e^{j\omega}) = G(e^{j\omega}) \]
From Table 5.2, we obtain 
\[ X(e^{j\omega}) = \frac{1}{1 - e^{-j\omega}} \]
Therefore, 
\[ H(e^{j\omega}) = (g[0] + g[1]e^{-j\omega}) + (1/2\pi)e^{j\omega} - g[0] - (1/2)e^{j\omega} = (1/2\pi)e^{j\omega} - g[0] - (1/2)e^{j\omega} \]
Clearly, $h[n]$ is a three-point sequence. We have 
\[ H(e^{j\omega}) = H[0] + H[1]e^{-j\omega} + H[2]e^{-2j\omega} \]

5.11. We know from the time expansion property (Table 5.1, Property 5.3.7) that 
\[ g[n] = a[n] \left( \frac{1}{2\pi} \right) \int_{-\pi}^{\pi} G(e^{j\omega}) e^{j\omega n} d\omega \]
Therefore, $G(\alpha\omega)$ is obtained by compressing $X(e^{j\omega})$ by a factor of $\alpha$. Since we know that $X(\alpha\omega)$ is periodic with a period of $2\pi$, we may conclude that $G(\alpha\omega)$ has a period which is $(1/\alpha)\pi = \pi$. Therefore, 
\[ G(\alpha\omega) = G(e^{j\alpha\omega}) \]
and $\alpha = \pi$.

5.12. Consider the signal 
\[ x[n] = \sin \left( \frac{\pi n}{m} \right) \]
From Table 2.2, we obtain the Fourier transform of $x[n]$ to be 
\[ X(e^{j\omega}) = \left\{ \begin{array}{ll} 1, & 0 \leq |\omega| \leq \frac{\pi}{m} \\ 0, & \frac{\pi}{m} < |\omega| \leq \pi \end{array} \right. \]

The plot of $X(e^{j\omega})$ is as shown in the Figure 5.12. Now consider the signal $x_2[n] = \frac{1}{\pi} [x[n]]^2$. Using the multiplication property (Table 5.1, Property 5.5), we obtain the Fourier transform of $x_2[n]$ to be 
\[ X_2(e^{j\omega}) = \frac{1}{\pi} \left( (1/\pi) X(e^{j\omega}) \right) \cdot X(e^{j\omega}) \]
This is plotted in the Figure 5.12.

From Figure 5.12 it is clear that $X_2(e^{j\omega})$ is zero for $|\omega| > \pi/2$. By using the convolution property (Table 5.1, Property 5.4), we note that 
\[ Y(e^{j\omega}) = X_2(e^{j\omega}) \left( \frac{1}{\pi} \sin \left( \frac{\pi n}{m} \right) \right) \]

and 
\[ H(\alpha e^{j\omega}) = h[0] + h[1]e^{-j\omega} + h[2]e^{-2j\omega} \]
We see that $H(e^{j\omega}) = H(\alpha e^{j\omega})$ only if $h[0] = 0$. We also have 
\[ H(\alpha e^{j\omega}) = h[0] + h[1]e^{-j\omega} + h[2]e^{-2j\omega} \]
Since we are also given that $Y(e^{j\omega/2}) = 1$, we have 
\[ h[0] - h[2] = 1 \]

Now note that 
\[ g[n] = h[0] + (1/4^n) w[n] \]
\[ x[2k] = \sum_{n=k}^{\infty} h[0] (1/4)^n w[n-k] \]
Evaluating this equation at $n = 2$, we have 
Since $h[0] = 0$, 
\[ \frac{1}{16} h[1] + h[2] = 0 \]
Solving equations (5.14-1) and (5.14-2), we obtain 
\[ h[0] = 1/17 \]
and 
\[ h[2] = \frac{1}{17} \]

Therefore, 
\[ h[n] = \frac{16}{17} d[n] - \frac{2}{17} d[n-2] \]

5.15. Consider $x[n] = \sin(\omega_0 n)/\sin(\omega_0/2)$, the Fourier transform $X(\alpha\omega)$ of $x[n]$. In this case, $X(\alpha\omega)$ is as shown in Figure 5.15. We note that the given signal $x[n]$ is $a[n] w[n]$. Therefore, the Fourier transform $Y(\alpha\omega)$ of $y[n]$ is 
\[ Y(x^n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) X(e^{-j\omega}) d\omega \]
Employing the approach used in Example 5.15, we can convert the above periodic convolution into an aperiodic signal by defining 
\[ \xi(\omega) = \left\{ \begin{array}{ll} X(e^{j\omega}), & 0 < \omega \leq \pi \\ 0, & \text{otherwise} \end{array} \right. \]
Then we may write

$$Y(\omega) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\omega) X(\omega + \eta) e^{-j\omega \eta} d\eta$$

This is the apodized convolution of the rectangular pulse \( X(\omega) \) shown in Figure 5.15 with the periodic square wave \( X(\omega) \). The result of this convolution is as shown in the Figure 5.15.

From the figure, it is clear that we require \( \omega \pm 2(\pi/2) \) to be \( \pi/2 \). Therefore, \( \omega = \pi/4 \).

5.16. We may write

$$X(\omega) = \frac{1}{2\pi} \int_{-\pi}^{\pi} G(\omega) Q(\omega(\omega + 2\pi)) d\eta$$

where \( \nu \) denotes apodized convolution. We may also rewrite this as a periodic convolution

$$X(\omega) = \frac{1}{2\pi} \int_{-\pi}^{\pi} G(\omega) Q(\omega(\omega + 2\pi)) d\eta$$

where

$$G(\omega) = \frac{1}{1 - e^{-j2\eta}}$$

and

$$Q(\omega) = \frac{\frac{1}{2} e^{-j\omega}}{1 - e^{-j2\eta}}$$

for \( 0 \leq \omega < 2\pi \).

(a) Taking the inverse Fourier transform of \( G(\omega) \) (see Table 5.2), we get \( g[n] = \{1/4\}^{n} a[n] \). Therefore, \( a[n] = 1 \).

(b) Taking the inverse Fourier transform of \( Q(\omega) \) (see Table 5.2), we get

$$q[n] = \frac{1}{2} e^{j\pi/2} + \frac{1}{2} e^{-j\pi/2}$$

This signal is periodic with a fundamental period of \( N = 4 \).

5.20. (a) Since the LTI system is causal and stable, a single input-output pair is sufficient to determine the frequency response of the system. In this case, the input is \( x[n] = (4/5)^{n} a[n] \) and the output is \( y[n] = a[n]/(4/5)^{n} a[n] \). The frequency response is given by

$$H(e^{j\omega}) = X(e^{j\omega})$$

where \( X(e^{j\omega}) \) and \( Y(e^{j\omega}) \) are the Fourier transforms of \( x[n] \) and \( y[n] \) respectively. Using Table 5.2, we have

$$x[n] = (4/5)^{n} a[n] \quad x[\omega] = \frac{1}{1 - e^{-j\pi/2}}$$

Using the differentiation in frequency property (Table 5.1, Property 5.3.8), we have

$$y[n] = (4/5)^{n} a[n] \quad y[\omega] = \frac{dX(\omega)}{d\omega} = \frac{(4/5)^{n} e^{j\pi/2}}{1 - e^{-j\pi/2}}$$

Therefore,

$$H(e^{j\omega}) = \frac{(4/5)^{-j\pi/2}}{1 - e^{-j\omega}}$$

(b) Since \( H(e^{j\omega}) = Y(e^{j\omega})/X(e^{j\omega}) \), we may write

$$Y(e^{j\omega}) = X(e^{j\omega}) \left[ \frac{1 - e^{-j\pi/2}}{4/5 e^{-j\pi/2}} \right].$$

Taking the inverse Fourier transform of both sides

$$y[n] = \frac{1}{5} x[n]$$

5.21. (a) The given signal is

$$x[n] = a[n] - 2^n a[2n] - a[n - 1] + 2^n a[n - 2] + a[n + 1] + 2^n a[n + 2]$$

Using the Fourier transform analysis eq. (5.9), we obtain

$$X(e^{j\omega}) = a e^{j\omega} + e^{j2\omega} + e^{-j\omega} + e^{-j2\omega}.$$

(b) Using the Fourier transform analysis eq. (5.9), we obtain

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} \left( \frac{1}{2} a e^{j\omega} - e^{j\omega} \right)$$

$$= \sum_{n=0}^{\infty} \left( \frac{1}{2} a e^{j\omega} + e^{j\omega} \right)$$

$$= a/2 + \frac{1}{e^{j\omega}}$$

(c) We can easily show that \( X(e^{j\omega}) \) is not conjugate symmetric. Therefore, \( a[n] \) is not real.

5.17. Using the duality property, we have

$$(-1)^{n} \frac{d^{n}}{d\omega^{n}} a[n] \rightarrow \frac{d^{n}}{d\omega^{n}} \left( \frac{1}{(1 - e^{-j\omega})^{n}} \right) = \frac{1}{(1 - e^{-j\omega})^{n+1}}$$

5.18. Knowing that

$$\left( \frac{1}{2 \pi} \right) \int_{-\pi}^{\pi} \frac{1 - e^{-j\omega}}{\cos(\omega/2)} d\omega = \frac{2}{2} = \frac{\pi}{2 \cos(\omega/2)}$$

we may use the Fourier transform analysis equation to write

$$\frac{1}{b - \pi \cos(\omega/2)} \int_{-\pi}^{\pi} \frac{2}{2} \left( \frac{b}{2} \right) e^{j\omega}$$

Putting \( \omega = -2\pi \) in this equation, and replacing the variable \( n \) by the variable \( k \)

$$\frac{1}{b - \pi \cos(\omega/2)} \int_{-\pi}^{\pi} \frac{2}{2} \left( \frac{b}{2} \right) e^{j\omega}$$

By comparing this with the continuous-time Fourier series synthesis equation, it is immediately apparent that \( a[n] = \left( \frac{1}{2} \right)^{n} \) are the Fourier series coefficients of the signal \( 1/(b - \pi \cos(\omega/2)) \).

5.19. (a) Taking the Fourier transform of both sides of the difference equation, we have

$$Y(\omega) = \frac{1}{2} e^{j\pi/2} + \frac{1}{2} e^{-j\pi/2} = X(\omega)$$

Therefore,

$$H(e^{j\omega}) = \frac{1}{1 - e^{-j\pi/2}}$$

(b) Using Partial fraction expansion,

$$H(e^{j\omega}) = \frac{3\pi}{1 - e^{-j\pi/2}}$$

Using Table 5.2, and taking the inverse Fourier transform, we obtain

$$x[n] = \left( \frac{1}{2} \right)^{n} a[n] + \frac{3}{2} \left( -\frac{1}{2} \right)^{n} a[n].$$

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(c) Using the Fourier transform analysis eq. (5.9), we obtain

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} \frac{1}{2} a e^{j\omega} e^{j2\omega}$$

$$= \sum_{n=0}^{\infty} \frac{1}{2} a e^{j\omega} e^{j2\omega}$$

$$= a e^{j\omega} + e^{j2\omega} + e^{-j\omega} + e^{-j2\omega}.$$}

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(b) The given signal is
\[ x[n] = \sin(3\pi n/3) + \cos(7\pi n/3) \]
\[ = -\sin(\pi n/3) + \cos(\pi n/3) \]
\[ = \frac{1}{\sqrt{2}}[e^{-j\pi n/3} + e^{j\pi n/3}] + \frac{1}{\sqrt{2}}[e^{-j\pi n/3} - e^{j\pi n/3}] \]

Therefore,
\[ X(e^{j\omega}) = -\frac{1}{\sqrt{2}}[\delta(\omega) - \pi(\omega)] - \delta(\omega - 3\pi) - \delta(\omega + 3\pi) \]

for \( 0 \leq |\omega| < 2\pi \).

(i) \( x[n] \) is periodic with period 6. The Fourier series coefficients of \( x[n] \) are given by
\[ a_k = \frac{1}{6} \sum_{n=0}^{5} x[n]e^{-j2\pi kn/6} \]
\[ = \frac{1}{6} \left[ e^{-j\pi n/3} - e^{j\pi n/3} \right] \]
\[ = \frac{1}{6} \left[ 1 - e^{-j6\pi n/6} \right] \]

Therefore, from the results of Section 5.2
\[ X(e^{j\omega}) = \sum_{n=0}^{5} \frac{1}{6} \left[ 1 - e^{-j6\pi n/6} \right] \frac{e^{-j\omega n}}{2\pi} \]

Using the differentiation in frequency property of the Fourier transform,
\[ n \left( \frac{1}{2} \right) e^{-j\omega n} \frac{d}{d\omega} e^{-j\omega n} \]

Therefore,
\[ x[n] = n \left( \frac{1}{2} \right) e^{-j\omega n} \frac{d}{d\omega} e^{-j\omega n} \]

(k) We have
\[ x_1[n] = \frac{\sin(\pi n/3)}{\pi n} \int_{-\infty}^{\infty} X(e^{j\omega}) e^{j\pi n/3} \frac{d\omega}{2\pi} \]

This is the Fourier transform of a periodic signal with fundamental frequency \( \pi/3 \).

Therefore, its fundamental period is 6. Also, the Fourier series coefficients of this signal are \( a_k = (-1)^k \).

Also,
\[ x_2[n] = \cos(3\pi n/2) + \cos(4\pi n/2) \]
\[ X_2(e^{j\omega}) = \pi \delta(\omega - \pi/2) + \delta(\omega + \pi/2) \]

is the range \( 0 \leq |\omega| < 2\pi \). Therefore, if \( x[n] = x_1[n] + x_2[n] \), then
\[ X(e^{j\omega}) = \text{periodic} \]

Using the mechanics of periodic convolution demonstrated in Example 5.15, we obtain in the range \( 0 \leq |\omega| < 2\pi \),
\[ X(e^{j\omega}) = \begin{cases} \frac{1}{\pi}, & 0 < |\omega| < \pi \\ 0, & \text{otherwise} \end{cases} \]

5.22. (a) Using the Fourier transform synthesis eq. (5.5.8), we obtain
\[ x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega n} d\omega \]

(b) Comparing the given Fourier transform with the analysis eq. (5.5.8), we obtain
\[ x[n] = \delta[n] - 2\delta[n - 1] + 2\delta[n - 2] - 4\delta[n - 3] - 6\delta[n - 4] + 6\delta[n - 6] \]

(c) Using the Fourier transform synthesis eq. (5.5.6), we obtain
\[ x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{j\omega n} X(e^{j\omega}) d\omega \]

(d) The given Fourier transform is
\[ X(e^{j\omega}) = \cos^2 \omega + \sin^2 (\omega/2) \]

Comparing the given Fourier transform with the analysis eq. (5.5.6), we obtain
\[ x[n] = \frac{1}{2}[\delta[n] + 2\delta[n - 1] + \frac{1}{2}\delta[n + 1] - \frac{1}{2}\delta[n - 1] - \frac{1}{2}\delta[n + 1] + \frac{1}{4}\delta[n + 3] + \frac{1}{4}\delta[n - 3]) \]

(e) We have from eq. (5.5.6)
\[ 2\pi x[0] = \int_{-\pi}^{\pi} X(e^{j\omega}) d\omega \]

Therefore,
\[ \int_{-\pi}^{\pi} X(e^{j\omega}) d\omega = 0 \]

(f) We have from eq. (5.5.6)
\[ X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n} \]

(e) From Table 5.1, we have
\[ X(e^{j\omega}) \] as in Figure 5.22.

(f) From Parseval's theorem we have
\[ \int_{-\pi}^{\pi} |X(e^{j\omega})|^2 d\omega = 2\pi \sum_{n=-\infty}^{\infty} |x[n]|^2 = 28 \]

(5) Using the differentiation in frequency property of the Fourier transform, we obtain
\[ x[n] = \frac{n}{\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega n} d\omega \]

Again using Parseval's theorem, we obtain
\[ \int_{-\pi}^{\pi} |X(e^{j\omega})|^2 d\omega = 2\pi \sum_{n=-\infty}^{\infty} |x[n]|^2 = 216\pi \]

5.24. (1) For \( \Re[X(e^{j\omega})] \) to be zero, the signal must be real and even. Only signals (b) and (d) are real and even.

(2) For \( \Im[X(e^{j\omega})] \) to be zero, the signal must be real and even. Only signals (d) and (h) are real and even.
(3) Assume \( Y(e^{j\omega}) = e^{j\omega}X(e^{j\omega}) \). Using the time shifting property of the Fourier transform, we have \( y[n] = x[n - m] \). If \( Y(e^{j\omega}) \) is real, then \( y[n] \) is real and even (assuming that \( x[n] \) is even). Therefore, \( y[n] \) has to be symmetric about \( n \). This is true only for signals (a), (b), (c), (e), (f), and (h).

(4) Since \( \int_{-\infty}^{\infty} X(e^{j\omega}) d\omega = 2\pi \delta[0] \), the given condition is satisfied only if \( x[0] = 0 \). This is true for signals (b), (c), (f), (b), and (i).

(5) \( X(e^{j\omega}) \) is always periodic with period 2\( \pi \). Therefore, all signals satisfy this condition.

(6) Since \( X(e^{j\omega}) = \sum_{\infty}^{\infty} x[n] e^{-j\omega n} \), the given condition is satisfied only if the samples of the signal add up to zero. This is true for signals (a), (g), and (i).

5.25. If the inverse Fourier transform of \( X(e^{j\omega}) \) is \( x[n] \), then

\[
x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega n} d\omega
\]

Therefore, the inverse Fourier transform of \( B(\omega) \) is \(-j 2\pi x[n] \). Also, the inverse Fourier transform of \( A(\omega)e^{j\omega} \) is \( x[n + 1] \). Therefore, the time function corresponding to the inverse Fourier transform of \( B(\omega) + A(\omega)e^{j\omega} \) will be \( x[n + 1] - j 2\pi x[n] \). This is shown in Figure 5.25.

5.26. (a) We may express \( X(e^{j\omega}) \) as

\[
X(e^{j\omega}) = \mathcal{F}\{x[n]\} = \mathcal{F}\{x[n + 1]\} + \mathcal{F}\{x[n + 1]e^{-j2\pi n/4}\}
\]

Therefore,

\[
x[n] = \mathcal{F}\{x[n]\} \left[ 1 + e^{j2\pi n/4} + e^{-j2\pi n/4}\right].
\]

5.27. (a) Let \( W(e^{j\omega}) \) be the periodic convolution of \( X(e^{j\omega}) \) with \( P(e^{j\omega}) \). The Fourier transforms are sketched in Figure 5.27.

(b) The Fourier transform of \( Y(e^{j\omega}) \) is \( W(e^{j\omega})P(e^{j\omega}) \). The LTI system with unit sample response \( \delta[n] \) is an ideal lowpass filter with cutoff frequency \( \pi/2 \). Therefore, \( Y(e^{j\omega}) \) for each choice of \( P[n] \) are as shown in Figure 5.27. Therefore, \( y[n] \) in each case is:

(i) \( y[n] = 0 \)

(ii) \( y[n] = \left\lfloor \frac{\sin(\pi n/2)}{\pi n/2} \right\rfloor \cdot \cos(\pi n/2) \)

(iii) \( y[n] = \left\lfloor \frac{\sin(\pi n/2)}{\pi n/2} \right\rfloor \cdot \cos(\pi n/2) \)

(iv) \( y[n] = \left\lfloor \frac{\sin(\pi n/2)}{\pi n/2} \right\rfloor \cdot \cos(\pi n/2) \)

(v) \( y[n] = \left\lfloor \frac{\sin(\pi n/2)}{\pi n/2} \right\rfloor \cdot \cos(\pi n/2) \)

5.28. Let

\[
\frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega})Y(e^{j\omega}) d\omega = 1 + e^{-j\pi} = Y\left(0^+\right).
\]

Taking the inverse Fourier transform of the above equation, we obtain

\[
y[n]y[n] = 0 [n] + 0 [n - 1] = y[n].
\]

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(i) We have \( X(e^{j\omega}) = \frac{1}{1 - e^{-j\pi}} \).

Therefore,

\[
Y(e^{j\omega}) = \frac{\frac{1}{1 - e^{-j\pi}}} {1 - \frac{1}{1 - e^{-j\pi}}} = \frac{1}{1 - \frac{1}{1 - e^{-j\pi}}}
\]

Taking the inverse Fourier transform, we obtain

\[
y[n] = 3 \left(\frac{1}{2}\right)^n u[n] - 2 \left(\frac{1}{2}\right)^n u[n].
\]

(ii) We have \( X(e^{j\omega}) = \frac{1}{1 - e^{-j\pi}} \).

Therefore,

\[
Y(e^{j\omega}) = \frac{1}{1 - \frac{1}{1 - e^{-j\pi}}} = \frac{1 - e^{-j\pi}}{1 - \frac{1}{1 - e^{-j\pi}}}
\]

Taking the inverse Fourier transform, we obtain

\[
y[n] = 4 \left(\frac{1}{2}\right)^n u[n] - 2 \left(\frac{1}{2}\right)^n u[n] - 2 \left(\frac{1}{2}\right)^n u[n] - 2 \left(\frac{1}{2}\right)^n u[n]
\]

(iii) We have \( X(e^{j\omega}) = 2\pi \sum_{k=0}^{\infty} \delta(\omega - 2k + 1) \).

Therefore,

\[
Y(e^{j\omega}) = 2\pi \sum_{k=0}^{\infty} \delta(\omega - 2k + 1) \left[ \frac{1}{1 - \frac{1}{1 - e^{-j\pi}}} \right]
\]

Taking the inverse Fourier transform, we obtain

\[
y[n] = \frac{3}{2} (-1)^n \cdot 2\pi \sum_{k=0}^{\infty} \delta(\omega - 2k + 1) \cdot 2
\]

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(b) Given
\[ h[n] = \frac{1}{2} \left( \frac{1}{2} e^{-j\pi/2} \right)^n u[n] + \frac{1}{2} \left( \frac{1}{2} e^{j\pi/2} \right)^n u[n]. \]
we obtain
\[ H(e^{j\omega}) = \frac{1}{2} + \frac{1}{2} e^{-j\pi/2} e^{-j\omega} + \frac{1}{2} e^{j\pi/2} e^{j\omega}. \]
(i) We have
\[ X(e^{j\omega}) = \frac{1}{1 - e^{-j\pi/2}}. \]
Therefore,
\[ Y(e^{j\omega}) = X(e^{j\omega}) \cdot H(e^{j\omega}) = \frac{1}{1 - e^{-j\pi/2}} \cdot \frac{1}{2} + \frac{1}{2} e^{-j\pi/2} e^{-j\omega} + \frac{1}{2} e^{j\pi/2} e^{j\omega}. \]
for \( \omega = -j\pi/2 \), \( B = 1/2 \), and \( C = j/2 \). Therefore,
\[ y[n] = \frac{1}{2} \left[ \frac{1}{2} \left( \frac{1}{2} e^{-j\pi/2} \right)^n + \frac{1}{2} \left( \frac{1}{2} e^{j\pi/2} \right)^n \right] = \frac{1}{2} \left( \delta[n] + \frac{1}{2} \left( \frac{1}{2} e^{-j\pi/2} \right)^n + \frac{1}{2} \left( \frac{1}{2} e^{j\pi/2} \right)^n \right) u[n]. \]
(ii) In this case,
\[ y[n] = \cos \left( \frac{\pi n}{2} \right) \delta[n] + \frac{1}{2} \left( \frac{1}{2} e^{-j\pi/2} \right)^n u[n]. \]
\[ x[n] = X(e^{j\omega}) \cdot H(e^{j\omega}) = \frac{1}{1 - e^{-j\pi/2}}. \]
\[ Y(e^{j\omega}) = \frac{1}{1 - e^{-j\pi/2}} \cdot \frac{1}{2} + \frac{1}{2} e^{-j\pi/2} e^{-j\omega} + \frac{1}{2} e^{j\pi/2} e^{j\omega}. \]
\[ y[n] = \frac{1}{2} \left[ \frac{1}{2} \left( \frac{1}{2} e^{-j\pi/2} \right)^n + \frac{1}{2} \left( \frac{1}{2} e^{j\pi/2} \right)^n \right] u[n]. \]
\[ X(e^{j\omega}) = \frac{1}{2} + \frac{1}{2} e^{-j\pi/2} e^{-j\omega} + \frac{1}{2} e^{j\pi/2} e^{j\omega}. \]
\[ Y(e^{j\omega}) = \frac{1}{2} \left[ \frac{1}{2} \left( \frac{1}{2} e^{-j\pi/2} \right)^n + \frac{1}{2} \left( \frac{1}{2} e^{j\pi/2} \right)^n \right] u[n]. \]
\[ Y(e^{j\omega}) = \frac{1}{2} \left[ \frac{1}{2} \left( \frac{1}{2} e^{-j\pi/2} \right)^n + \frac{1}{2} \left( \frac{1}{2} e^{j\pi/2} \right)^n \right] u[n]. \]
\[ Y(e^{j\omega}) = \frac{1}{2} \left[ \frac{1}{2} \left( \frac{1}{2} e^{-j\pi/2} \right)^n + \frac{1}{2} \left( \frac{1}{2} e^{j\pi/2} \right)^n \right] u[n]. \]
\[ Y(e^{j\omega}) = \frac{1}{2} \left[ \frac{1}{2} \left( \frac{1}{2} e^{-j\pi/2} \right)^n + \frac{1}{2} \left( \frac{1}{2} e^{j\pi/2} \right)^n \right] u[n]. \]
\[ Y(e^{j\omega}) = \frac{1}{2} \left[ \frac{1}{2} \left( \frac{1}{2} e^{-j\pi/2} \right)^n + \frac{1}{2} \left( \frac{1}{2} e^{j\pi/2} \right)^n \right] u[n]. \]
\[ Y(e^{j\omega}) = \frac{1}{2} \left[ \frac{1}{2} \left( \frac{1}{2} e^{-j\pi/2} \right)^n + \frac{1}{2} \left( \frac{1}{2} e^{j\pi/2} \right)^n \right] u[n]. \]
\[ Y(e^{j\omega}) = \frac{1}{2} \left[ \frac{1}{2} \left( \frac{1}{2} e^{-j\pi/2} \right)^n + \frac{1}{2} \left( \frac{1}{2} e^{j\pi/2} \right)^n \right] u[n]. \]
\[ Y(e^{j\omega}) = \frac{1}{2} \left[ \frac{1}{2} \left( \frac{1}{2} e^{-j\pi/2} \right)^n + \frac{1}{2} \left( \frac{1}{2} e^{j\pi/2} \right)^n \right] u[n]. \]
\[ Y(e^{j\omega}) = \frac{1}{2} \left[ \frac{1}{2} \left( \frac{1}{2} e^{-j\pi/2} \right)^n + \frac{1}{2} \left( \frac{1}{2} e^{j\pi/2} \right)^n \right] u[n]. \]
\[ Y(e^{j\omega}) = \frac{1}{2} \left[ \frac{1}{2} \left( \frac{1}{2} e^{-j\pi/2} \right)^n + \frac{1}{2} \left( \frac{1}{2} e^{j\pi/2} \right)^n \right] u[n]. \]
\[ Y(e^{j\omega}) = \frac{1}{2} \left[ \frac{1}{2} \left( \frac{1}{2} e^{-j\pi/2} \right)^n + \frac{1}{2} \left( \frac{1}{2} e^{j\pi/2} \right)^n \right] u[n]. \]
\[ Y(e^{j\omega}) = \frac{1}{2} \left[ \frac{1}{2} \left( \frac{1}{2} e^{-j\pi/2} \right)^n + \frac{1}{2} \left( \frac{1}{2} e^{j\pi/2} \right)^n \right] u[n]. \]
\[ Y(e^{j\omega}) = \frac{1}{2} \left[ \frac{1}{2} \left( \frac{1}{2} e^{-j\pi/2} \right)^n + \frac{1}{2} \left( \frac{1}{2} e^{j\pi/2} \right)^n \right] u[n]. \]
\[ Y(e^{j\omega}) = \frac{1}{2} \left[ \frac{1}{2} \left( \frac{1}{2} e^{-j\pi/2} \right)^n + \frac{1}{2} \left( \frac{1}{2} e^{j\pi/2} \right)^n \right] u[n]. \]
\[ Y(e^{j\omega}) = \frac{1}{2} \left[ \frac{1}{2} \left( \frac{1}{2} e^{-j\pi/2} \right)^n + \frac{1}{2} \left( \frac{1}{2} e^{j\pi/2} \right)^n \right] u[n]. \]
\[ Y(e^{j\omega}) = \frac{1}{2} \left[ \frac{1}{2} \left( \frac{1}{2} e^{-j\pi/2} \right)^n + \frac{1}{2} \left( \frac{1}{2} e^{j\pi/2} \right)^n \right] u[n]. \]
\[ Y(e^{j\omega}) = \frac{1}{2} \left[ \frac{1}{2} \left( \frac{1}{2} e^{-j\pi/2} \right)^n + \frac{1}{2} \left( \frac{1}{2} e^{j\pi/2} \right)^n \right] u[n]. \]
\[ Y(e^{j\omega}) = \frac{1}{2} \left[ \frac{1}{2} \left( \frac{1}{2} e^{-j\pi/2} \right)^n + \frac{1}{2} \left( \frac{1}{2} e^{j\pi/2} \right)^n \right] u[n]. \]
\[ Y(e^{j\omega}) = \frac{1}{2} \left[ \frac{1}{2} \left( \frac{1}{2} e^{-j\pi/2} \right)^n + \frac{1}{2} \left( \frac{1}{2} e^{j\pi/2} \right)^n \right] u[n]. \]
In the case
\[ X(e^{j\omega}) = 1 + \frac{1}{2} e^{-j\omega}. \]
Therefore,\[ Y(e^{j\omega}) = 1. \]
Taking the inverse Fourier transform, we obtain\[ y[n] = \delta[n]. \]

(iv) In this case
\[ X(e^{j\omega}) = 1 - \frac{1}{2} e^{-j\omega}. \]
Therefore,
\[
Y(e^{j\omega}) = \left[ 1 - \frac{1}{2} e^{-j\omega} \right] \left[ \frac{1}{1 + \frac{1}{2} e^{-j\omega}} \right] \\
= -\frac{1}{2} + \frac{1}{2} e^{-j\omega}. \\
\]
Taking the inverse Fourier transform, we obtain
\[ y[n] = -\frac{1}{2} \delta[n] + 2 \left( -\frac{1}{2} \right)^n u[n]. \]

(c) (i) We have
\[
Y(e^{j\omega}) = \left[ \frac{1}{1 - \frac{1}{2} e^{-j\omega}} \right] \left[ \frac{1}{1 + \frac{1}{2} e^{-j\omega}} \right] \\
= \frac{1}{1 - \frac{1}{2} e^{-j\omega}} \frac{1}{1 + \frac{1}{2} e^{-j\omega}} \\
= \frac{1}{1 - \frac{1}{4} \cos \omega}. \\
\]
Taking the inverse Fourier transform, we obtain
\[ y[n] = (n+1) \left( \frac{1}{2} \right)^n u[n] - \frac{1}{2} \left( \frac{1}{2} \right)^n u[n-1]. \]

(ii) We have
\[
Y(e^{j\omega}) = \left[ \frac{1}{1 + \frac{1}{2} e^{-j\omega}} \right] \left[ \frac{1}{1 + \frac{1}{2} e^{-j\omega}} \right] \\
= \frac{1}{1 + \frac{1}{4} \cos \omega}. \\
\]
Taking the inverse Fourier transform, we obtain
\[ y[n] = \left( \frac{1}{2} \right)^n u[n]. \]

5.34. (a) Since the two systems are cascaded, the frequency response of the overall system is
\[ H(e^{j\omega}) = H_1(e^{j\omega})H_2(e^{j\omega}). \]
Therefore, the Fourier transforms of the input and output of the overall system are related by
\[ Y(e^{j\omega}) = \frac{2 - e^{-j\omega}}{1 + e^{-j\omega}}. \]
Cross-multiplying and taking the inverse Fourier transform, we get
\[ y[n] + y[n-1] = 2u[n] - u[n-1]. \]

(b) We may rewrite the overall frequency response as
\[ H(e^{j\omega}) = \frac{4/3}{1 + \frac{1}{3} e^{-j\omega}}. \]
Taking the inverse Fourier transform we get
\[ y[n] = \frac{2}{3} \left( \frac{1}{2} \right)^n u[n] + \frac{1 + \sqrt{3}}{3} \left( \frac{1}{2} \right)^{n-1} u[n-1] + \frac{1 - \sqrt{3}}{3} \left( \frac{1}{2} \right)^{n-2} u[n-2]. \]

This is as sketched in Figure 5.35.

5.36. (a) The frequency responses are related by the following expression:
\[ G(e^{j\omega}) = H(e^{j\omega})^{-1}. \]
(b) (i) Here, \( H(e^{j\omega}) = 1 - \frac{1}{2} e^{-j\omega} \). Therefore, \( G(e^{j\omega}) = (1 + \frac{1}{4} e^{-j\omega})^{-1} \) and \( y[n] = (\frac{1}{2})^n u[n] \). Since
\[ G(e^{j\omega}) = Y(e^{j\omega}) \]
the difference equation relating the input \( x[n] \) and output \( y[n] \) is
\[ y[n] - \frac{1}{2} y[n-1] = x[n]. \]

(ii) Here, \( H(e^{j\omega}) = \frac{1}{1 + \frac{1}{2} e^{-j\omega}} \). Therefore, \( G(e^{j\omega}) = 1 + \frac{1}{2} e^{-j\omega} \) and \( y[n] = \frac{1}{2} \delta[n] + \delta[n-1] \). Since
\[ G(e^{j\omega}) = Y(e^{j\omega}) \]
the difference equation relating the input \( x[n] \) and output \( y[n] \) is
\[ y[n] = x[n] + \frac{1}{2} y[n-1]. \]

(iii) Here, \( H(e^{j\omega}) = (1 + \frac{1}{2} e^{-j\omega})/(1 + \frac{1}{4} e^{-j\omega}) \). Therefore, \( G(e^{j\omega}) = (1 + \frac{1}{4} e^{-j\omega})/(1 + \frac{1}{2} e^{-j\omega}) \) and \( y[n] = x[n] + \frac{1}{2} y[n-1] \). Since
\[ G(e^{j\omega}) = Y(e^{j\omega}) \]
the difference equation relating the input \( x[n] \) and output \( y[n] \) is
\[ y[n] - \frac{1}{2} y[n-1] = x[n] + \frac{1}{2} y[n-1]. \]

(iv) Here, \( H(e^{j\omega}) = (1 + \frac{1}{2} e^{-j\omega})/(1 + \frac{1}{4} e^{-j\omega}) \). Therefore, \( G(e^{j\omega}) = (1 + \frac{1}{4} e^{-j\omega})/(1 + \frac{1}{2} e^{-j\omega}) \) and \( y[n] = \frac{1}{2} \delta[n] + \delta[n-1] \). Since
\[ G(e^{j\omega}) = Y(e^{j\omega}) \]
the difference equation relating the input \( x[n] \) and output \( y[n] \) is
\[ y[n] - \frac{1}{2} y[n-1] = x[n]. \]
(v) Here, \( H(e^{j\omega}) = 1 + \frac{1}{1 + e^{j\omega}} \). Therefore, \( G(e^{j\omega}) = 1 + \frac{1}{1 + e^{j\omega}} \).

\[
G(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = 1 + \frac{1}{1 + e^{j\omega}}.
\]

The difference equation relating the input \( x[n] \) and output \( y[n] \) is

\[
y[n] - y[n-1] = 2y[n] + \frac{1}{2}x[n] - \frac{1}{2}x[n-2].
\]

(vi) Here, \( H(e^{j\omega}) = 1 + \frac{1}{1 + e^{j\omega}} \). Therefore, \( G(e^{j\omega}) = 1 + \frac{1}{1 + e^{j\omega}} \).

\[
G(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = 1 + \frac{1}{1 + e^{j\omega}}.
\]

Since

\[
y[n] = 2y[n] + \frac{1}{2}x[n] - \frac{1}{2}x[n-2],
\]

and the difference equation relating the input \( x[n] \) and output \( y[n] \) is

\[
y[n] = x[n] + \frac{1}{2}x[n] - \frac{1}{2}x[n-2].
\]

(c) The frequency response of the given system is

\[
H(e^{j\omega}) = \frac{1}{1 + e^{j\omega}}.
\]

The frequency response of the inverse system is

\[
G(e^{j\omega}) = 1 + \frac{1}{1 + e^{j\omega}}.
\]

Therefore,

\[
y[n] = \left(\frac{1}{2}\right)^{n+1} x[n+1] + \left(\frac{1}{2}\right)^{n} x[n+1] + \frac{1}{2} \left(\frac{1}{2}\right)^{n-1} x[n-1].
\]

Clearly, \( g[n] \) is not a causal impulse response.

If we delay this impulse response by \( 1 \) sample, then it becomes causal. Furthermore, the output of the inverse system will be \( x[n-1] \). The impulse response of this causal system is

\[
y[n] = 2y[n-1] - y[n].
\]

Since \( x[n] \) is real, \( X(e^{j\omega}) = X^*(e^{j\omega}) \). Therefore,

\[
x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} \text{Re}\{X(e^{j\omega})\} e^{-j\omega n} \, d\omega + \frac{1}{2\pi} \int_{-\pi}^{\pi} \text{Im}\{X(e^{j\omega})\} e^{j\omega n} \, d\omega
\]

\[
= \frac{1}{\pi} \int_{-\pi}^{\pi} \text{Re}\{X(e^{j\omega})\} \cos(\omega n) \, d\omega + \frac{1}{\pi} \int_{-\pi}^{\pi} \text{Im}\{X(e^{j\omega})\} \sin(\omega n) \, d\omega.
\]

Therefore,

\[
H(e^{j\omega}) = \frac{1}{\pi} \text{Re}\{X(e^{j\omega})\} \cos(\omega n), \quad \text{and} \quad \frac{1}{\pi} \text{Im}\{X(e^{j\omega})\} \sin(\omega n).
\]

5.39. Let \( y[n] = x[n] \cdot \delta[n] \). Then

\[
Y(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] \cdot \delta[n] e^{j\omega n} = \sum_{n=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} x[k] \cdot \delta[n-k] e^{j\omega n} = \sum_{k=-\infty}^{\infty} x[k] \sum_{n=-\infty}^{\infty} \delta[n-k] e^{j\omega n} = \sum_{k=-\infty}^{\infty} x[k] e^{-j\omega k} H(e^{j\omega}) = X(e^{j\omega}) H(e^{j\omega}) = H(e^{j\omega}) X(e^{j\omega}).
\]

5.40. Let \( y[n] = x[n] \cdot h[n] \). Then using the convolution sum

\[
y[n] = \sum_{k=-\infty}^{\infty} x[k] h[n-k]. \tag{5.40-1}
\]

Using the convolution property of the Fourier transform,

\[
y[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) H(e^{j\omega}) \, d\omega \tag{5.40-2}
\]

Now let \( h[n] = \delta[n] \). Then \( H(e^{j\omega}) = X(e^{j\omega}) \). Substituting in the right-hand sides of equations (5.40-1) and (5.40-2), and equating them,

\[
\sum_{n=-\infty}^{\infty} x[n] h[n] \delta[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) H(e^{j\omega}) \, d\omega.
\]

Therefore,

\[
X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n} = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n} = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n} H(e^{j\omega}) = X(e^{j\omega}) H(e^{j\omega}) = H(e^{j\omega}) X(e^{j\omega}).
\]

5.41. (a) The Fourier transform of the signal \( x[n] \) is

\[
X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n} = \sum_{n=-N+1}^{N-1} x[n] e^{-j\omega n}.
\]

Therefore,

\[
X(e^{j\omega}) = \sum_{n=-N+1}^{N-1} x[n] e^{-j\omega n} = \sum_{n=-N+1}^{N-1} x[n] e^{-j\omega n} = \sum_{n=-N+1}^{N-1} x[n] e^{-j\omega n} = \sum_{n=-N+1}^{N-1} x[n] e^{-j\omega n}.
\]

Now, we may write the expression for the FT coefficients of \( x[n] \) as

\[
a_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} \sum_{n=-N+1}^{N-1} x[n] e^{-j\omega n} \, d\omega.
\]

(Because \( x[n] = \delta[n] \) in the range \( n_0 \leq n \leq n_0 + N - 1 \). Comparing the above equation with eq. (5.41-1), we get

\[
a_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) \, d\omega.
\]

(b) (i) From the given information,

\[
X(e^{j\omega}) = 1 + e^{j\omega} + e^{-j\omega} = e^{j\omega} \left(1 + e^{j\omega} + e^{-j\omega}\right) = e^{j\omega} (2 \cos(\omega/2) + \cos(\omega/2)) = 2 \cos(\omega/2) + \cos(\omega/2).
\]

(ii) From part (a),

\[
a_n = \frac{1}{2\pi} X(e^{j\omega}) \, d\omega = \frac{1}{2\pi} \left(2 \cos(\omega/2) + \cos(\omega/2)\right) = \frac{1}{\pi} \left(2 \cos(\omega/2) + \cos(\omega/2)\right) = \frac{1}{\pi} \left(2 \cos(\omega/2) + \cos(\omega/2)\right).
\]

5.42. (a) \( P(e^{j\omega}) = 2\pi a_1 \) for \( |\omega| < \pi \). This is as shown in Figure 5.42. 199
(b) From the multiplication property of the Fourier transform we have
\[ G(\omega) = \int_{-\infty}^{\infty} X(x) e^{-j\omega x} dx = \int_{-\infty}^{\infty} x(\Phi(x)) e^{-j\omega x} dx \]
\[ = \frac{1}{2\pi} \int_{-\infty}^{\infty} x^{\prime}(\Phi(\omega)) e^{-j\omega x} dx \]
\[ = X(e^{j\omega}) \]

5.43. (a) Using the frequency shift and linearity properties,
\[ V(\omega') = \frac{1}{2} [X(\omega') + X(\omega')] \]
(b) Let \( a[n] = e^{j\omega n} \). Then
\[ Y(\omega') = \sum_{n=-\infty}^{\infty} e^{j\omega n} e^{-j\omega n} \]
Since the odd-indexed samples of \( x[n] \) are zero, we may put \( m = 2n \) in the above equation to get
\[ Y(\omega') = \sum_{n=-\infty}^{\infty} e^{j\omega n} e^{-j\omega n} \]

(5.44) The signal \( x[n] \) is as shown in Figure 5.44.
(i) Taking the inverse Fourier transform, the signal \( x[n] \) is
\[ x[n] = x[n + 1] = e^{j\omega n + 1} \]
\[ x[n] = e^{j\omega n} \]
\[ \therefore x[n] = e^{j\omega n} \]

5.45. From the power transform analysis equation,
\[ X(\omega') = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n} \]
\[ \therefore \]

From the differentiation in frequency property,
\[ \frac{d}{dt} x[n] \]
\[ \therefore \]

Therefore, we have shown that the result is valid for \( r = 1 \) and \( r = 2 \). Let us assume that it is also true for \( r = 3 \). We will now attempt to prove that the result is true for \( r = 3 \). We have
\[ s_{n+1}[n] = \frac{(n + r - 2)^2}{(n + r - 2)} x[n] \]
\[ \therefore x[n] = \frac{1}{(1 - a^2)^2} \]
\[ x[n] = \frac{1}{(1 - a^2)^2} \]
We know from Problem 5.43 that the inverse Fourier transform of \(X(\omega)\) is the sequence \(x[n] = |x| + e^{\omega x}X(n)/2\). The even-indexed samples of \(x[n]\) are identical to the even-indexed samples of \(x[n]\). The odd-indexed samples of \(x[n]\) are zero. If \(X(\omega) = X(\omega)\), then \(x[n] = x[n]\). This implies that the even-indexed samples of \(x[n]\) are zero. Consequently, \(x[n]\) does not necessarily have to be an impulse. Therefore, the given statement is false.

(d) From Table 5.1, we know that the inverse Fourier transform of \(X(\omega)\) is the time-expanded signal

\[
x[\frac{n}{2}] = \begin{cases} x[n], & n = 0, 2, 4, \ldots \\ 0, & \text{otherwise} \end{cases}
\]

If \(X(\omega) = X(\omega)\), then \(x[n] = x[n]\). This is possible only if \(x[n]\) is an impulse. Therefore, the given statement is true.

5.48. (a) Taking the Fourier transform of both equations and eliminating \(W(n)\), we obtain

\[
H(\omega) - Y(\omega)X(\omega) = \frac{3 - e^{-j\omega}}{1 - e^{-j\omega}}.
\]

Taking the inverse Fourier transform of the partial fraction expansion of the above expression, we obtain

\[
h[n] = 6 \left(\frac{1}{2}\right)^n u[n] - \left(\frac{1}{2}\right)^n u[n].
\]

(b) We know that

\[
H(\omega) = \frac{3 - e^{-j\omega}}{1 - e^{-j\omega}}.
\]

Cross-multiplying and taking the inverse Fourier transform, we obtain

\[
y[n] = \frac{3}{2} y[n - 1] - \frac{1}{2} y[n - 2] + y[n - 3] = \frac{3}{2} y[n - 1] - \frac{1}{2} y[n - 2].
\]

5.49. (a) Consider the signal \(y[n] = a[n] + b[n]\), where \(a[n]\) and \(b[n]\) are constants. Then, \(X(\omega) = a[1][e^{-j\omega}] + b[1][e^{-j\omega}]\). Also, let the response of the system to \(y[n] = 1\) be \(x[n] = y[n]\) and \(y[n] = \frac{1}{2} y[n] - y[n - 1]\), respectively. Substituting for \(X(\omega)\) in the expression given in the problem and simplifying we obtain \(Y(\omega) = a[1][e^{-j\omega}] + b[1][e^{-j\omega}]\). Therefore, the system is linear.

(c) Consider the signal \(x[n] = \delta[n - 1]\). Then, \(X(\omega) = e^{-j\omega}\). Let the response of the system to this signal be \(y[n] = 1\). From the given equation,

\[
y[n] = = 2X(\omega) + e^{-j\omega}Y(\omega) - \frac{dX(\omega)}{d\omega}.
\]

Therefore, the system is not time-invariant.

(b) From the given information,

\[
H(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{1 - e^{-j\omega}}{1 - e^{-j\omega}}.
\]

We now want to find \(X(\omega)\) when \(Y(\omega) = 1 + j \sin(\omega)/1 + e^{-j\omega}\). From the above equation we obtain

\[
X(\omega) = -e^{-j\omega}(1 - 1) = -e^{-j\omega}.
\]

Taking the inverse Fourier transform of the partial fraction expansion of the above expressions, we obtain

\[
x[n] = \frac{3}{2}(\frac{1}{2})^n u[n - 1] + \frac{1}{2}(\frac{1}{2})^n u[n - 1] + \frac{1}{2}(\frac{1}{2})^n u[n - 1] = \frac{3}{2}(\frac{1}{2})^n u[n - 1].
\]

5.51. (a) Taking the Fourier transform of \(y[n]\) we obtain

\[
H(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{\frac{1}{2} e^{-j\omega}}{1 - e^{-j\omega}}.
\]

Cross-multiplying and taking the inverse Fourier transform, we obtain

\[
y[n] = \frac{1}{2} y[n - 1] - \frac{1}{2} y[n - 2] + y[n - 1] = \frac{1}{2} y[n - 1] - \frac{1}{2} y[n - 2].
\]

Let us assume the intermediate output \(y[n]\). (See Figure S5.51)

5.52. (a) Since \(h[n]\) is causal, the nonzero sample values of \(h[n]\) and \(x[n] = \delta[n-1]\) overlap only at \(n = 0\). Therefore,

\[
\sum \{h[n]\} = \frac{1}{2} \sum \{\delta[n]\} - \frac{1}{2} \sum \{\delta[n-1]\}.
\]

In other words,

\[
h[n] = 2 \sum \{\delta[n]\}, \quad n \geq 0
\]

Now note that if

\[\sum \{\delta[n]\} = \frac{1}{2} \delta[n] - \frac{1}{2} \delta[n-1],\]

then

\[x[n] = \frac{1}{2} \delta[n] + \frac{1}{2} \delta[n-1].\]

Therefore,

\[\sum \{h[n]\} = \frac{1}{2} \delta[n] - \frac{1}{2} \delta[n-1].\]

Taking the inverse Fourier transform of \(X(\omega)\), we obtain

\[Y(\omega) = \frac{1}{2} \delta[n] + \frac{1}{2} \delta[n-1].\]

(c) Since \(h[n]\) is causal, the nonzero sample values of \(h[n]\) and \(x[n] = \delta[n] - \delta[n-2]\) overlap only at \(n = 0\). Therefore,

\[\sum \{h[n]\} = \frac{1}{2} \delta[n] + \frac{1}{2} \delta[n-1].\]

Therefore,

\[\sum \{h[n]\} = \frac{1}{2} \delta[n] + \frac{1}{2} \delta[n-1].\]

Taking the inverse Fourier transform of \(X(\omega)\), we obtain

\[Y(\omega) = \frac{1}{2} \delta[n] + \frac{1}{2} \delta[n-1].\]

Therefore,

\[\sum \{h[n]\} = \frac{1}{2} \delta[n] + \frac{1}{2} \delta[n-1].\]
In other words,

\[ h[n] = \begin{cases} 2Q(d[n]), & n > 0 \\ \text{some value,} & n = 0 \\ 0, & n < 0 \end{cases} \quad (5.52-2) \]

Now note that if

\[ h[n] = \frac{1}{2} \frac{\sin \omega n}{\sin \omega} \]

then

\[ X_e(\omega) = X_f(\omega - N) \]

Clearly, we can recover \( X_e(\omega) \) from \( X_f(\omega) \). From \( X_e(\omega) \) we can use eq. (5.52-2) to recover \( h[n] \) (provided \( X_f(\omega) \) is given). Obviously, from \( h[n] \) we can once again obtain \( X_f(\omega) \). Therefore, the system is completely specified by \( X_f(\omega) \) and \( h[n] \).

(4) Let \( X_m(\omega) \) be equal to \( \sin \omega \). Then,

\[ X_m(\omega) = \frac{1}{2} \frac{\sin \omega n}{\sin \omega} - \frac{1}{2} \frac{\sin \omega (n+1)}{\sin \omega} \]

Therefore,

\[ h[n] = h[0] \delta[n] + 4n \]

We may choose two different values for \( h[0] \) (say 1 and 2) to obtain two different systems whose frequency responses have imaginary parts equal to \( \sin \omega \).

5.53. (a) The analysis equation of the Fourier transform is

\[ X(\omega) = \sum_{n=0}^{N-1} x[n] e^{-j \omega n} \]

Comparing with eq. (5.53-2), we have

\[ X[k] = \frac{1}{N} X(\omega = 2\pi k/N) \]

(b) From the figure we obtain

\[ X_1(\omega) = 1 - e^{-j \omega} + 2e^{-j 2\omega} \]

and

\[ X_2(\omega) = -e^{-j \omega} + e^{j \omega} + 2e^{-j \omega} + 2e^{-j \omega} = 1 - e^{-j \omega} + 2e^{-j \omega} \]

Now,

\[ X_1(\omega = 2\pi k/N) = 1 - e^{-j \omega} + 2e^{-j \omega} \]

and

\[ X_2(\omega = 2\pi k/N) = 1 - e^{-j \omega} + 2e^{-j \omega} = X_1(\omega = 2\pi k/N) \]

5.55. (a) From Table 5.2, we have

\[ X_1(\omega) = 2 \sum_{n=0}^{N-1} \delta(n) e^{-j \omega n} \]

(1) When \( M = 1 \), \( P(\omega) = e^{j \pi} + 2 + e^{-j \pi} = 1 + 2 \cos \omega \).

(2) When \( M = 10 \), we may use Table 5.2 to find that

\[ P(\omega) = \sin(2\pi/7)/\omega/7 \]

(b) The plots are as shown in Figure 5.55.

\[ \text{Figure 5.55} \]

(c) We have \( W_1(\omega) = \frac{\sin(2\pi/7)}{\omega/7} \). The plots are as shown in Figure 5.55.

(d) The plots are as shown Figure 5.55.

5.54. (a) From eq. (5.54-1) it is clear that to compute \( \hat{X}[k] \) for one particular value of \( k \), we need to perform \( N \) complex multiplications. Therefore, in order to compute \( \hat{X}[k] \) for \( N \) different values of \( k \), we need to perform \( N(N + 1) \) complex multiplications.

(b) (i) Since \( y[n] = 2^n \), we have \( \hat{y}[0] = 2^n, \hat{y}[1] = 2^n, \ldots, \hat{y}[N-1] = 2^n \). Since \( \hat{y}[n] \) is nonzero only in the range \( 0 \leq n \leq N-1 \), \( \hat{y}[n] \) is nonzero only in the range \( 0 \leq n \leq N-1 \).

Similarly, since \( g[n] = 2^n \), we have \( \hat{g}[0] = 2^n, \hat{g}[1] = 2^n, \ldots, \hat{g}[N/2 - 1] = 2^n \). Since \( \hat{g}[n] \) is nonzero only in the range \( 0 \leq n \leq N-1 \), \( \hat{g}[n] \) is nonzero only in the range \( 0 \leq n \leq N/2 - 1 \).

(ii) We may rewrite eq. (5.54-1) as

\[ \hat{X}[k] = \frac{1}{N} \sum_{n=0}^{N-1} y[n] W_k^n \]

Since \( W_k^n = W_k^* \), we may rewrite the above equation as

\[ \hat{X}[k] = \frac{1}{N} \sum_{n=0}^{N-1} \overline{y[n]} W_k^n + \frac{1}{N} \sum_{n=0}^{N/2 - 1} \overline{y[n+1]} W_k^n \]

We have

\[ \hat{X}[k] = \frac{1}{N} \sum_{n=0}^{N-1} |y[n]|^2 \]

Similarly,

\[ \hat{G}[k] = \hat{G}[N/2 - k] = \overline{\hat{G}[k]} \]

(c) Since \( \hat{F}[k] \) is a N/2 point DFT, we may use an approach similar to the one in part (a) to show that we need \( N/2 \) complex multiplications to compute it. Similarly we may show that the computation of \( \hat{F}[k] \) requires \( N/2 \) multiplications. From eq. (5.54-1), it is clear that we need \( N^2/4 \) complex multiplications to compute \( X[k] \).

5.56. (a) From eq. (5.56-1) we have

\[ X(\omega, \phi) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} x[m, n] e^{j \omega n} e^{j \phi m} \]

(1) When \( \phi = 0 \),

\[ X(\omega, 0) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} x[m, n] e^{j \omega n} e^{j \phi m} = \sum_{m=-\infty}^{\infty} X[m, \omega] e^{j \phi m} \]

Therefore, we may write

\[ X(\omega, \phi) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\omega, \phi) e^{-j \phi m} d\phi \]

From this we obtain

\[ x[m, n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\omega, \phi) e^{j \phi m} e^{-j \phi n} d\phi \]

(2) We may easily show that

\[ X(\omega, \phi) = A(\omega) B(\phi) \]

We use the result of the previous part in many of the problems of this chapter.

(i) \( X(\omega, \phi) = e^{j \phi} A(\omega) \)

(ii) \( X(\omega, \phi) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\omega, \phi) e^{-j \phi m} d\phi \)

(iii) \( X(\omega, \phi) = \int_{-\pi}^{\pi} X(\omega, \phi) e^{j \phi m} e^{-j \phi n} d\phi \)

From the definition of the 2D Fourier transform we obtain

\[ X(\omega, \phi) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{j \omega \phi} e^{-j \phi} d\phi \]

From the definition of the 2D Fourier transform we obtain

\[ X(\omega, \phi) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{j \omega \phi} e^{-j \phi} d\phi \]

(3) \( X(\omega, \phi) = A(\omega) B(\phi) \)

(3) \( X(\omega, \phi) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\omega, \phi) e^{j \phi m} e^{-j \phi n} d\phi \)

(3) \( X(\omega, \phi) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\omega, \phi) e^{j \phi m} e^{-j \phi n} d\phi \)